P. S. NOVIKOV

ELEMENTS OF MATHEMATICAL LOGIC

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ELEMENTS OF
MATHEMATICAL LOGIC

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A. J. Lohwater, Consulting Editor

ELEMENTS OF MATHEMATICAL LOGIC

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PREFACE

The central part which mathematical logic plays in modern mathematics, and its value in the training of the young mathematician, have received increasingly wide recognition in recent years, but the development of undergraduate courses in modern mathematical logic has been hampered by the lack of a first-rate introductory text. Novikov's book is remarkable for the help which it affords the reader, by its lucid and painstaking exposition and by its wealth of illustrative examples; it is a book which makes no compromise with rigour and yet keeps the student's needs in mind.

After a preliminary discussion of the nature of formal systems and finitist metalogic, the opening chapters present sentence and predicate logic both as formal axiomatic systems and, informally, in set theoretic interpretations. Notions of consistency, axiom independence and completeness are carefully discussed, and the deduction theorems, and Gödel's completeness theorem for predicate logic, are proved in detail.

Arithmetic is presented as an extension of predicate logic obtained by adjoining as new axioms the defining equations of all primitive recursive functions together with axioms of equality and substitution, order axioms, and the axiom of induction. Gödel's incompleteness theorem for arithmetic is discussed, but not proved.

In the final chapter, Novikov introduces an interesting new reduction process and applies it to prove the consistency of restricted arithmetic (arithmetic without induction) and the independence of the axiom of induction of the remaining axioms of arithmetic.

P. S. Novikov is one of Russia's foremost mathematicians. Before the war he was well known for his work in measure theory and the theory of sets. From set theory he turned to mathematical logic and in 1955 he solved one of the great outstanding problems on the border of logic and algebra, the word problem for groups, and constructed a finitely generated group which admits no recursive decision procedure for word-equality; recently, the experience gained in his study of the word problem led Novikov to the solution of the famous Burnside problem, in which he disproved Burnside's conjecture by constructing an infinite group which is periodic and generated by 72 generators.

R. L. GOODSTEIN

FOREWORD

The recent intensive development of mathematical logic has been accompanied by a broadening of its role in mathematics.

The analysis of the foundations of mathematics is still one of the fundamental problems of mathematical logic but mathematical logic has already gone beyond the bounds of this problem and has exerted essential influence on mathematics itself. Out of its ideas there arose a precise definition of the concept of algorithm which allowed the solution of many problems which, without it, would have remained in principle unsolvable. Apparatus which originated in mathematical logic found application in problems of the construction of calculating machines and automatic schemata.

In the Russian language there are two systematic textbooks on mathematical logic: one of them is *Principles of Mathematical Logic* by Hilbert and Ackermann, published in 1948, and the other is Kleene's monograph *Introduction to Metamathematics*, published in 1957. [These are translations of Hilbert-Ackermann: *Grundzüge der theoretischen Logik*, Springer, Berlin, 1938, second edition and Kleene: *Introduction to Metamathematics*, Noordhoff, Groningen, 1952.] In the first-named book, the discussion is somewhat concise—thus possibly presenting difficulties on a first contact with the subject-matter. The second-named book is rather of an encyclopedic character and is suitable for those who are already acquainted with the fundamental ideas of mathematical logic.

At the present time, the number of persons who are interested in mathematical logic is constantly increasing. Courses, the presentation of which require knowledge of the elementary ideas of mathematical logic, are given at universities and pedagogical institutes. The presence of a suitable textbook would be very useful in connection with these courses. In the present book an attempt is made to give a discussion of the fundamentals of mathematical logic in the most accessible form possible. The first five chapters of the book, constituting its basic content, are devoted to this problem. The last (i.e. the sixth) chapter is of a more special character and is no longer so elementary. In this chapter, methods of the theory of proof are considered, by means of which certain problems in mathematical logic which arise in the basic text of the book are solved.

Of the most important attainments of modern mathematical logic, one must mention Gödel's theorem on the incompleteness of deductive systems,

and the concept of general recursive functions. Kleene's monograph gives an account of these and also other problems in mathematical logic.

I express my profound gratitude to S. I. Adyan, A. S. Yesenin-Vol'pin and A. A. Lunts, who were of great help to me and spent much effort in preparing the manuscript of this book.

P. S. NOVIKOV

CONTENTS

INTRODUCTION

In modern mathematics, the so-called axiomatic method has received wide acceptance. Its source is to be found in the discovery of non-Euclidean geometry by Lobachevsky. Up to the present time, the axiomatic method has undergone extensive evolution, having come into contact with other ideas and generating not only new methods but also new principles of thinking in mathematics as well as in the physical sciences. The development of the axiomatic method can be divided into two stages. The first extends from Lobachevsky to the works of Hilbert on the foundations of mathematics and the second from these works of Hilbert up to the present time. [In Russian translation, these works were published in 1948 (see D. Hilbert, *Foundations of Geometry*, Supplements VI-X).] This second stage constitutes a combination of ideas, proceeding from geometry, with the study, which developed parallel with it, known under the name of "symbolic" or "mathematical" logic. As a result, there arose a new discipline which retained the name mathematical logic.

Before discussing mathematical logic itself, we shall consider briefly the state of the mathematical method which precedes it, and we shall strive to explain, if only in very general outline, the origin of this method and the problems which confront it. The essence of the axiomatic method consists in the definition of mathematical objects and the relations among them in some particular way. We shall assume that, studying a system of objects, we use well-defined terms which express the properties of these objects and the relations among them. We do not define the objects themselves nor these properties and relations, but we state a number of well-defined propositions which must be satisfied by them. Obviously, these propositions pick out from all possible systems of objects (and properties and relations among them) those systems for which they are true.

Thus, the propositions affirmed can be considered as definitions of systems of objects of a well-defined class, together with properties and relations among these objects.

We now consider a simple example, which we shall also encounter frequently in the sequel. Suppose a system of objects is prescribed, which we shall denote by the letters of the Latin alphabet, and suppose that a relation between them is established which is expressed by the term "pre-

cedes". Without defining either the objects, or the relation "precedes", we state the following propositions about them:

1. No object precedes itself.
2. If x precedes y and y precedes z, then x precedes z.

It is easily seen that there exist systems of objects with relations among them such that if the phrase "x precedes y" is understood to be a given relation, then our propositions turn out to be true.

Suppose, for example, that the objects $x, y, \ldots$ are people and that the relation between x and y is "x is older than y". In this case, if "x precedes y" means "x is older than y", then propositions 1 and 2 are true.

Or again, if the objects are real numbers and the relation "x precedes y" means "x is less than y", then here also propositions 1 and 2, obviously, are satisfied.

Systems of objects with one relation, for which propositions 1 and 2 are satisfied, form a well-defined class, and propositions 1 and 2 can be considered as the definition of the systems of this class. Propositions by means of which we thus isolate the set of objects are called *axioms*. If for any set of objects and their properties and relations certain axioms are true, then we say that the given set of objects satisfies this system of axioms or that it is an interpretation of the given system of axioms.

Performing logical deductions from the axioms, we shall obtain propositions which are true for an arbitrary system of objects which satisfy the given axioms.

A more significant example of an axiomatic definition is the system of axioms of geometry. We consider a system of objects, and subdivide them into "points", "straight lines" and "planes" and we use such expressions as "the point lies in a straight line", "the straight line lies in a plane", "the point a lies between the points b and c", and others, which express relations between objects of the system. Furthermore, when we use these terms, we shall not immediately endow them with the sense of spatial relations, but instead we shall point out a certain system of axioms for them. This can be done in various ways, but there exists a completely well-defined system of axioms, called "the system of axioms of Euclidean geometry". This system was suggested by Hilbert. We shall not introduce these axioms here, but they can be found in various books on the foundations of geometry. In these axioms are made explicit all those presuppositions which, in explicit or implicit form, are used in the proof of theorems of Euclidean geometry. Thus, consequences deduced from these axioms express adequately the properties of Euclidean space, an intuitive representation of which drawn from direct experience has long existed in the human mind.

It is clear that the correspondence between axioms and the objects of

reality always have an approximate character. If we, for example, pose the problem whether real physical space satisfies the axioms of Euclidean geometry, we must first give physical definitions of the geometric terms which are contained in the axioms, such as : "point", "straight line", "plane", and others. In other words, we must indicate those physical circumstances which correspond to these terms. After this, the axioms transform into physical assertions which can be subjected to experimental verification. After such verification, we can vouch for the truth of our assertions with that degree of precision which the measuring instruments guarantee.

In considering an arbitrary system of axioms, a number of problems arise which can be solved only with the aid of interpretations. One of these problems is that of the consistency of a system of axioms. We should always be convinced that by performing all possible deductions from given systems of axioms we do not arrive at a contradiction, i.e. we do not deduce incompatible propositions. The occurrence of a contradiction would mean that no system of objects can satisfy the system of axioms under consideration and, thus, these axioms do not describe anything. The consistency of a system of axioms can be proved by constructing some concrete interpretation of this system. It should be noted that in the pre-Hilbert axiomatic method this was the only means for proving consistency.

The situation is analogous for the problem of the independence of axioms. An axiom is said to be independent in a given system of axioms if it is *not* deducible from the remaining axioms of this system. To prove the independence of an axiom it is sufficient to find a system of objects which satisfies all the axioms except the one being investigated and which does not satisfy the latter. In other words, to prove the independence of an axiom one must find an interpretation of the system of axioms obtained from the system under consideration by replacing the axiom being investigated by its negation. Therefore, in order to make use of a system of axioms, it is necessary to have in advance objects, properties and relations which can serve as a concrete interpretation of this system of axioms.

Interpretations of systems of axioms derive from a number of mathematical concepts. *The most powerful source of interpretations for all possible systems of axioms is the theory of sets.*

We shall not go into much detail in the discussion of set theory. We shall only point out the most general features—namely the objects with which it deals.

The initial objects are natural numbers. From the set of natural numbers one can, with the aid of set-theoretic principles, construct new sets and functions. We shall point out certain fundamental principles for the construction of sets.

1. If a set of objects is given, one can isolate a subset of it by means of a

precisely formulated criterion. For example, from the set of all natural numbers one can isolate a subset—the set of all prime numbers.

2. If we have some aggregate of sets, then it is possible to obtain a new set by combining all the elements of all these sets.

3. For every set one can form the set of all its subsets.

4. We shall assume that, in virtue of some criterion, to each element of a set E one can assign some element of a set G. Such a correspondence is called a function. We say that the function under consideration is defined on the set E and that it assumes values from the set G. Functions are also objects with which one can construct sets. In particular, one can form the set of all functions which are defined on E and assume values in G.

The principles listed above do not exhaust all possible means for constructing set-theoretic objects. But for our further discussion we may limit ourselves to the means described here. With the aid of set-theoretic principles, starting from the set of natural numbers considered as the initial set, we can construct all existing mathematical concepts. But then this also exhausts all interpretations for systems of axioms.

The question arises: Is the theory of sets a completely safe foundation for the axiomatic method? To what extent can we be sure of the consistency of the theory of sets itself?

This discipline, which arose at the end of the last century, developed rapidly and exerted an enormous influence on mathematics and it had fundamental significance in problems of the foundations of mathematics. But at the very beginning of the development of set theory it was noted that the unrestricted use of the concepts created by it leads to contradictions. This situation did not stop the development of set theory, as in those fields in which its concepts are employed, no contradiction in fact arose. And neither did a further analysis of the foundations of set theory yield any satisfactory basis for believing that, at least within the framework of the factual utilization of the ideas of set theory, no contradiction can arise. Thus, the assertion of the absence of contradiction within the bounds of existing set-theoretical constructions is an empirical conclusion for which there are no sufficiently weighty foundations. As a result, one must admit that, despite the very successful service rendered to the axiomatic method by the theory of sets, the foundations on which it itself rests are not satisfactory. Further criticism turned its attention to one essential peculiarity of the theory of sets, or rather to a peculiarity of mathematical thought in general, but occurring more clearly in the development of set theory. We are dealing here with the idea of infinity, which is one of the most fundamental elements of mathematical thought. In the mathematics of antiquity, great caution was shown with respect to infinity. A logical analysis of the concepts connected with infinity was made to a degree which is well known. This was reflected in the appearance of the well-known antinomies: "Achilles and the tortoise",

"the flying arrow", "infinite divisibility", and others. Caution was expressed by requirements of great rigor in the use of infinity in mathematical reasoning. Modern mathematical analysis, in the very beginning of its origin under the influence of the requirements of the natural sciences and technology, began to turn to infinity with much greater freedom and less rigor. Thanks to this, it achieved the possibility of rapid and extensive development and played an enormous role in the most varied branches of science and practice. In this connection—and this is the important thing—the difficulties connected with the idea of infinity were of influence in that each time they arose, they aroused criticism of the corresponding concepts of analysis. This critical tendency was most apparent in its last stage—namely, in the works of Kronecker, Shatunovsky, Borel, Lusin and Brouwer. The form of infinity, which lies at the base of set-theoretical representations, received the name of "actual infinity". Before making this concept precise, we shall attempt to form a picture of it with the aid of a crude and logically imprecise description. By the phrase "actual infinity" we understand an infinite set whose construction is completed and whose elements are represented simultaneously. We shall, for example, deal with actual infinity if we read off the entire sequence of natural numbers completely. Now if it were possible to introduce an arbitrary infinite set of acts which are strictly separated from one another, then there would not be any mathematical problems. Each problem would be solved by a direct verification of all possible cases. The idealized character of the concept of actual infinity is perfectly clear. The construction of an infinite number of individual objects, accomplished by an infinite number of acts, is unrealizable not only in virtue of the insufficiency of practical means but because it is fundamentally impossible by any means. Moreover, mathematical thinking widely uses this idealization—for instance, in representing a geometric figure as an infinite set of points, an interval of time as an infinite set of moments, motion as an infinite set of individual positions of the moving body, and so on.

The idea of actual infinity is involved in the generalization to infinity of certain logical principles which are absolutely beyond question in the finite domain. One of such principles is, for example, the well-known law of the excluded middle which is formulated as follows: let A be a proposition and $\bar{A}$ its negation. Then one of the propositions A and $\bar{A}$ is true. Let us assume that A is a proposition about the objects of an infinite set—for example, about the set of all natural numbers. If we could perform an infinite number of acts of verification, they would show that either A or $\bar{A}$ is true. The assumption that it is possible to say of an arbitrary proposition A, that it or its negation is true, is a partial substitute for the hypothesis of the possibility of an infinite number of acts of verification. The absolute interpretation of the term "existence" in applications to infinite sets also plays the same role in the theory of sets. For the theory of sets, theorems of pure "existence" are

characterized by proofs that such-and-such an object exists without exhibiting or constructing this object itself. Such proofs are often connected with the use of the law of the excluded middle.

We now consider an example of such a proof. We construct an infinite sequence of non-negative integers associated with the expansion of the number $\pi = 3{\cdot}14 \ldots$ in a decimal fraction. Every term a_n of the sequence is determined—depending on the nth digit in the expansion of π. If the nth digit in the expansion equals zero, then the number a_n corresponding to it is also set equal to zero. The numbers $a_1, a_2, \ldots,$ up to the first zero, are set equal to unity. After the first zero occurs (or several zeros in succession), the following numbers, corresponding to digits which are not equal to zero—right up to a new zero—are set equal to two. Then, after the appearance of a zero (or several zeros in succession), all numbers corresponding to digits which are not equal to zero, up to a new zero, are set equal to three, and so on. In general, if the numbers in the sequence which precede a zero (or a group of several zeros in succession) equal k, then the numbers in the sequence following this zero to the next zero equal $k+1$. We thus obtain a sequence of (approximately) the following form:

$$1, \ldots, 1, 0, 2, \ldots, 2, 0, \ldots, k, \ldots, k, 0, \ldots, 0, k+1, \ldots.$$

We shall prove that there exists a number which is repeated in this sequence an infinite number of times. In fact, in the expansion of the number π, there are either a finite or an infinite number of zeros. In the first case, we find the number a_n in the sequence with the largest index n such that $a_n = 0$. After this a_n, all numbers of the sequence are equal and, consequently, the number a_{n+1} repeats an infinite number of times. If the expansion of the number π contains an infinite number of digits equal to zero, zero is repeated an infinite number of times in the sequence. Thus, the existence of a number which repeats an infinite number of times in the constructed sequence has been established. However, we cannot tell what this number is since it is unknown whether the expansion of π contains a finite or infinite number of zeros and there appears to be no approach to the solution of such a problem. Thus we have here an example of a proof of the existence of some object which we have no means of exhibiting. This situation is explicitly connected with actual infinity. In considering only finite structures, if the existence of some object is proved, one can always find this object by a factual testing of all possible cases.

At first glance it may appear in the consideration of such examples that there is no correspondence of the idea of actual infinity with reality. In fact, this only emphasizes the limitational, approximative character of the correspondence of the mathematical representations under consideration and the real world. The idea of actual infinity within defined intelligible bounds, as well as many other ideal concepts, can therefore be utilized. This

explanation does not, of course, exhaust all difficulties connected with the concept of infinity. If we pose the question of whether or not the idea of actual infinity leads to a contradiction, it is difficult at the present time to give any answer except, perhaps, that up to this time no contradiction has been found.

But even if we accept the hypothesis that the satisfaction of the concept of actual infinity in the theory of sets does not lead to contradiction, this does not completely remove the difficulties of the theory of sets. Another question arises: Can we be sure that every mathematical problem is solvable with the aid of the theory of sets. The hypothesis of the solvability of an arbitrary problem by means of the theory of sets (in the absence of any hypothesis of inconsistency) appears to us to be unlikely. But, in any case, for a proof of the truth of this conjecture it is necessary to have the means of proving it.

One can try to resolve such problems by means of the axiomatic method. To this end it is necessary to find all premises from which deductions are made in set theory, formulate them in the form of axioms and try to solve the problem of *consistency* for the system of axioms obtained. In this case, the *decision* problem reduces to the problem of independence since to prove that a problem is not decidable in set theory we must establish the fact that the corresponding assertion and its negation are not deducible from the given axioms. It is possible to find a system of axioms, described within the known bounds of set theory, but the solution of the problems of consistency and independence for such a system of axioms comes up against essential difficulties. We cannot use the method of interpretation in the problems under consideration. The reason for this is that we find interpretations for arbitrary systems of axioms within the bounds of set theory and in virtue of this the consistency of the theory of sets itself must already be assumed.

To avoid the difficulties thus created, Hilbert proposed a new way of viewing the problems under consideration. Hilbert's ideas were of decisive importance in the problem of the foundations of mathematics and marked the beginning of a new era in the development of the axiomatic method. We pose the following question: To what extent is it necessary for the solution of problems of consistency and independence to use the method of interpretation exclusively? Is it impossible to do without interpretations in the solution of these problems?

Let us assume that some system of axioms is given, for which we wish to solve the problem of consistency. As we have already pointed out, an inconsistent system is a system in which some proposition A together with its negation, which we denote by $\bar{A}$, can be deduced.

Therefore, to prove that a system of axioms is inconsistent, it suffices to find a proposition A for which a deduction, from the given system of axioms, of both A itself and its negation are realizable. In order to prove the consistency of a system of axioms, it is sufficient to show that no matter what

proposition is taken, there does not exist a deduction from the axioms of both it and its negation. If we could describe all possible propositions which can be expressed within a given system of axioms and furthermore describe all possible methods of deduction, then perhaps we would succeed in deriving directly from this description all possible instances of the simultaneous deduction of any proposition and its negation. In exactly the same way one could also formulate problems of independence. To prove that a proposition A does not depend on the axioms $A_1, \ldots, A_n$ would then mean to deduce from the prescribed description of deductions the proof that there cannot be a proof of the proposition A from the axioms $A_1, A_2, \ldots, A_n$. It may be that the description of all possible propositions and deductions does not require the introduction of the very powerful methods of set theory and does not include such doubtful concepts as, for example, actual infinity. The description of all possible forms of deduction proved to be completely attainable and the way for it had, in essence, already been prepared by the preceding development of symbolic logic. However, in problems of consistency, new difficulties were encountered; despite these, the situation was rescued from stagnation, and a new tendency in the axiomatic method has begun to develop. The difficulties which arise, although serious, are of such a nature that the possibility of their removal is apparent and they are in fact being gradually removed.

Thus, we must first of all have a set of completely reliable concepts and principles of thought (in every case with respect to consistency), in the framework of which we may perform all subsequent constructions. In order to exclude from the basic set of concepts all doubtful elements of set-theoretic thinking, it is natural to attempt to choose it in the most restricted way possible. Moreover, it is not possible to exclude infinity completely from consideration; but, on the other hand, it is completely possible to destroy its "actual" character. The concept of actual infinity figured in philosophy long before the appearance of the works, described here, on the foundations of mathematics and consideration was given to the idea of an infinity of another type which received the name of "potential infinity" as a concept which was opposed to the former. The meaning of this concept is that one considers an infinite set of realizable possibilities. Each of them, taken separately, is realizable—and an arbitrary finite number of these possibilities is also realizable—but all, taken together, are not realizable. Let us consider an example. We shall assume that the construction of an integer is realized if some set of objects containing the given number of elements is exhibited. For every given integer it is basically possible to picture the corresponding set. This can also be done for an arbitrary finite number of integers, but it is impossible to realize a representation of all integers.

There are no intelligent reasons for doubting the legitimacy of the use of the concept of potential infinity at the present state of science. Not only

mathematics, but also the exact sciences, cannot do without this sort of infinity. In any case, criticism of the foundations of mathematics raises no objections to this concept. The idea of potential infinity lies at the base of Hilbert's conception.

We shall stop to consider the basic principles of Hilbert's study in somewhat more detail. We pose two problems.

1. Find the set of concepts and principles which do not contain doubtful aspects of set-theoretical thought.

2. Within the framework of such a set of concepts, formulate the question of consistency and independence for an arbitrary system of axioms—in particular, the axioms of set theory.

If we succeed in this way in solving the problems of consistency and independence, then we obtain a possibility of forming a foundation for the use of the idea of actual infinity and of elucidating the bounds within which this is possible. We shall begin with the first problem—the construction of systems of concepts and principles which satisfy the requirements set down.

We shall consider systems consisting of a finite number of elements, certain specified properties of these elements, and relations among them. It is immaterial to us what these elements, properties and relations are. We require only that all these elements be legibly distinguishable from one another—together with the properties and the relations between them. For every specified property and relation it must be precisely determined for which elements it holds and for which it does not. Let us consider some examples.

1. Let

$$a2c45e3$$

be a row of letters and numerals. We note two properties of the elements of this row: (1) to be a numeral; (2) to be a letter, and one relation between the elements x and y of the row—the element x in the row precedes y.

2. Suppose that in some collection of spheres there are three white and four black spheres, where the diameters of all spheres are different, and suppose that if we arrange these spheres in order of increasing diameters, then their colours are distributed in the following way: white, black, white, black, black, white, black. Take the specified properties of this system of sphere to be the colours of the spheres and the specified relation between the spheres x, y to be that the diameter of the sphere x is less than the diameter of y.

3. The system consists of metallic rings. A single relation among the rings is specified, that of one ring being fitted onto another.

When considering a system, we shall abstract from the qualitative nature of its elements its specified properties and relations. Two systems will be said to be isomorphic if a one-to-one correspondence can be established

between their elements whereby the specified properties (relations) of one system correspond to the specified properties (relations) of the other system. It is easy to see that the first two systems introduced above are isomorphic. We shall not distinguish between isomorphic systems. This means that we consider essentially not concrete systems but rather schemes of systems. Every scheme determines an entire class of mutually isomorphic systems and every system of this class can represent a scheme if we perform for it only such operations as are applicable to an arbitrary system of a given class.

It is possible to find a representative for every scheme. To this end it suffices to take an arbitrary set with the corresponding number of elements:

$$x_1, x_2, \ldots, x_n,$$

select from it subsets:

$$A_1, A_2, \ldots, A_{m_1},$$

respectively, for every specified property; then, select sets, corresponding to the number of binary relations, whose elements are ordered pairs (x_i, x_j):

$$B_1, B_2, \ldots, B_{m_2};$$

further, select sets whose elements are ordered triples (x_i, x_j, x_k), and so on; finally, select sets whose elements are ordered groups of s elements:

$$U_1, U_2, \ldots, U_{m_s}. \tag{*}$$

The last sets (*) will correspond to relations among s elements of the system. The group of symbols: $x_1, x_2, \ldots, x_n$ with the specified properties A_i and relations $B_1, \ldots, U_{m_s}$ among the elements can serve as a representative of the scheme.

In the sequel, we shall consider schemes containing only binary specified relations, i.e. relations between pairs of elements. In many cases we shall restrict ourselves to schemes which are rows of symbols where the specified relation is the relation of precedence in a row, and the elements are the symbols themselves. For example,

$$\alpha\beta\gamma.$$

Here, the specified relation among elements is expressed in the following way: α precedes β, α precedes γ, β precedes γ. For all remaining pairs, precedence does not hold. We shall also make use of rows in which one and the same symbol can be repeated several times. For example,

$$aabacddc.$$

This means that we introduce implicitly special specified properties: "to be the element a", "to be the element b", and so forth. Two elements which possess the same specified property of this type are said to be identical.

In the sequel, for brevity, we shall use the term "configuration" instead of the phrase "system scheme".

We now consider examples of configurations.

1. Every natural number n can be considered as a configuration containing n elements; specified properties and relations are not defined in this configuration. All representations of such a configuration with the same number of elements are isomorphic and, therefore, such a configuration can serve as a definition of the concept of "number of elements".

2. A configuration contains n elements, and among them there is specified a unique relation expressed by the phrase "x precedes y". Conditions, or axioms, defining this relation are the order axioms considered above. In this connection, to the two order axioms we adjoin still a third:

if x is different from y, then either "x precedes y" or "y precedes x".

It is easily seen that a representative of such a configuration is the row

$$x_1x_2 \ldots x_n,$$

in which the relation "x precedes y" denotes "x is situated to the left of y in the given row".

Further, we define constructive classes of the configuration and constructive operations on the configurations, in which connection these definitions must be subject to the following requirements. For every element of the constructive class, the fact that this element belongs to this class is established, starting with the definition, by means of a basically realizable set of operations. Suppose the constructive operation $T(A_1, A_2, \ldots, A_n)$ assigns some configuration B to an arbitrarily prescribed set of configurations $A_1, A_2, \ldots, A_n$. The definition of the operation $T(A_1, A_2, \ldots, A_n)$ must always yield a basically realizable method of constructing the configuration B when the configurations $A_1, A_2, \ldots, A_n$ are given.

We note that in some cases the constructive operation $T(A_1, A_2, \ldots, A_n)$ can be defined not for arbitrary configurations $A_1, A_2, \ldots, A_n$ but for configurations belonging to a well-defined class. For distinct variables A_i and A_j the classes of configurations for which the operation T is defined may turn out to be distinct.

Simple examples of constructive classes of configurations are the class of natural numbers n and also the class of sequences or rows considered above.

For the latter class, we define the operation $S(A, B)$ consisting of conjoining the row B to the row A. The elements of the configuration $S(A, B)$ are all elements of the configurations A and B. For an arbitrary pair of elements of the configuration $S(A, B)$ an order relation is defined. We agree that every element belonging to A precedes every element belonging to B. For pairs of elements x and y belonging to A (or to B) we retain the same order relations "x precedes y" which are established in the configurations A or B respectively. It is easily seen that if A and B represent the rows

$$x_1x_2 \ldots x_n$$

and

$$y_1y_2 \ldots y_m,$$

respectively, then the configuration $S(A, B)$ can be represented by the row

$$x_1x_2 \dots x_ny_1y_2 \dots y_m.$$

We shall write the operation $S(A, B)$, described above, in the form AB. The constructive character of this operation is perfectly obvious. One can, with the aid of this operation, starting with one-element configurations, obtain the class of all possible rows.

Another example of a constructive operation is the operation

$$R(A, a, B),$$

which consists in replacing the element a wherever it occurs in the row A by the row B.

The operation $R(A, a, B)$, which is called the "substitution operation", will frequently be encountered in the sequel. We will use it for the definition of a somewhat more complicated operation. We define the operation

$$T(A, a, B, n),$$

where a is an arbitrary element, A and B are arbitrary rows, and n is a natural number as follows:

$T(A, a, B, 1)$ coincides with $R(A, a, B)$.

$T(A, a, B, 2)$ is the result of replacing the element a wherever it occurs in the row $T(A, a, B, 1)$ by the row B, i.e. it is

$$R[T(A, a, B, 1), a, B].$$

In an analogous manner, $T(A, a, B, 3)$ is determined by $T(A, a, B, 2)$, and so forth. To determine $T(A, a, B, n)$, for arbitrary $n > 1$, one can write the following recurrence formula:

$$T(A, a, B, n) = R[T(A, a, B, n-1), a, B].$$

The constructive character of this operation is also easily observed from the definition.

Configurations, constructive classes of configurations and constructive operations form that aggregate of concepts which we assume as the basis of all further constructions. Moreover, we exclude the use of concepts which are not reducible to them. However, in order to exclude completely the use of infinity in the actual form, we must also restrict the means for reasoning about these concepts. Classes of configurations which we have already introduced are, generally speaking, infinite, and the use for them of such logical principles as the law of the excluded middle deprives these infinities of their potential character. We shall describe those logical and mathematical principles whose use is permitted. Within the bounds of consideration of one, or of an arbitrary finite number, of configurations for all discussions performed only in terms of the elements of the configuration, properties and relations among these elements, we permit all logical and mathematical means without any restrictions. In all other discussions, in

view of the fact that in them infinity can already occur, we eliminate from the general logical principles the "law of the excluded middle". All the remaining logical principles will be retained. In particular, the "inconsistency law" is allowed. This law, as is known, consists in the assertion of the impossibility of any proposition and its negation being true simultaneously. In virtue of this, certain forms of proof by contradiction find a place in the allowable discussions. The concept of "existence" will be used in the sense of "possibility of construction". Of the mathematical principles of proof, we shall keep the one called the "axiom of complete induction". The application of this principle is connected with constructive definitions. We shall assume that we have defined some set of objects by means of prescribing some initial objects and operations by the application of which one constructs an arbitrary object of the given set (as one defines, for instance, constructive classes). Then the principle of complete induction in applications to the set is formulated in the following way.

If any assertion holds for the initial objects of the given set and if the validity of the assertion for the result of an arbitrary prescribed operation follows from the validity of this assertion for objects on which the given operation is performed, then the assertion holds for all objects of the given set.

This principle is applicable to the constructive classes described above as well as to constructive operations. Its legitimacy with respect to the idea of potential infinity is not subject to any doubt. The set of concepts and methods of reasoning we have described represent a well-defined system of thought. Arguments and constructions performed within the framework of this system will be called constructive or finitary, and the system as a whole will be referred to as Hilbert's finitism.

We proceed to the second problem—within the framework of the system introduced, utilizing its concepts and principles only, to formulate the problems of consistency and independence for arbitrary axiomatic systems. If we retained the previous notion of axioms seeking to find for them an interpretation within the concepts of finitism, then we would restrict the possibilities of solving the problems connected with the axioms since finitism turns out to be a very weak method of obtaining an interpretation for the very simplest systems of axioms. Hilbert suggested considering axioms from another point of view. Axioms are defined by propositions. But propositions, no matter what sense they may have, are always combinations of terms and perhaps symbols arranged among themselves in some connection. Performing logical deductions, we proceed from certain arrangements to others. The question arises whether it is possible to describe deductive operations on propositions in the form of a mechanism in the realm of a finitary system of thought. More precisely, the question is formulated as follows:

Is it possible to represent all propositions of any aggregate of problems interesting us in the form of a configuration and the rules of logic used in the form of constructive operations?

If this is so, then every system of axioms can be represented as a set of well-defined configurations and the conclusions deduced from them form a constructive class of configurations. It turns out that such a representation of propositions and logical conclusions is entirely possible.

We assume that we have defined some class of configurations which represent propositions and that in it there are indicated well-defined configurations which we call axioms. Suppose, moreover, that there are also indicated constructive operations which represent the logical operations of deduction. In this case, the entire system obtained will be called a formalism, a formal logical system, deductive calculus or simply a calculus. The terms "formalism" and "calculus" will always be used as synonyms. Arbitrary propositions of the formalism will be called formulae. Operations which represent logical inferences are called deduction rules. Axioms and formulae which are obtained from the axioms by applying the deduction rules are called formulae which are deducible in the given formalism (or in the given calculus). We shall sometimes call the deducible formulae true formulae in the given calculus. In the formalism, there can figure also configurations which are not formulae, but which occur in their construction and participate in the definition of formulae. The assertion that a given configuration is deducible will be called a formal theorem, and the actual construction of a deducible configuration by applying the rules of deduction will be called a formal deduction.

The consistency problem is formulated for calculi for which the concept of formal negation is defined in such a way that to every formula is assigned another formula called its negation. We shall call a formalism (or calculus) inconsistent if a formula together with its negation are deducible in it. The problem of the independence of the axioms of a formalism is posed in the question: Is a given axiom deducible in a formalism, which differs from the formalism being considered by the fact that this axiom is eliminated from the list of its axioms? Or, in brief: Is a given axiom deducible by means of the rules of the formalism from the remaining axioms?

Reasoning about a formalism which is restricted by the bounds of finitism is called *metalogic*. *One must strictly distinguish between informal deductions which are made in the proofs of various statements about calculi and formal deductions in the calculus which are operations on configurations and are considered only as such.* Symbols which do not belong to the set of those elements of which the configurations are formed and which are introduced to denote concepts of the calculus are frequently called *metalogical symbols.* We speak of *metalogical reasoning* in an analogous way.

We also note that the restrictions about which we spoke in the description

of Hilbert's finitism do not extend to the concepts and deductions within the calculi themselves. These restrictions (in particular, the restriction on the use of the law of excluded middle) apply only to the means of describing formalisms and reasoning about formalisms.

We now consider an example of a calculus. We shall first describe a propositional system in an informal way. In these propositions, we are dealing with numbers (it is immaterial what numbers they are—natural, real or complex). Every lower-case Latin letter represents an arbitrary number. We call those letters variables which take on numerical values. We shall consider the operations of addition and multiplication of numbers and the equality of numbers, denoting them in the usual way. We write out the initial equations which are true for all values occurring for their variables:

1. $a + b = b + a$,
2. $(a + b) + c = a + (b + c)$,
3. $ab = ba$,
4. $(ab)c = a(bc)$,
5. $(a + b)c = ac + bc$.

From these equations, one can deduce others by making use of the following two principles:

1. In a true equation, every variable may be replaced wherever it occurs by an arbitrary numerical expression containing arbitrary variables.

2. In any true equation, an arbitrary numerical expression may be replaced by an expression which is equal to it.

We shall now describe this system in the form of a formalism. We shall first describe the configurations corresponding to numerical expressions and equations. The basic store of elements of the configurations consists of the following symbols:

1. Lower case Latin letters $a, b, \ldots, x, y, \ldots$.
2. Pairs of brackets ( ).
3. The symbols $=$, $+$.

Configurations corresponding to numerical expressions will be called terms. They are defined in the following way.

Every lower case Latin letter is a term.

If α and β are terms, then $(\alpha + \beta)$ and $(\alpha\beta)$ are also terms.

A constructive class of rows, which we call terms, are defined by these conditions. In the definition which we just introduced, terms are configurations formed by numerical expressions in the form in which they are usually written with the only difference that compound terms are enclosed in parentheses—for example:

$$(a + b),\ ((a + b)c), \text{ and so forth.}$$

We shall agree, for the sake of brevity, not to write exterior brackets.

Equations are defined as rows of the form

$$\alpha = \beta,$$

where α and β are arbitrary terms.

Formulae in our formalism are equations. A term is not a formula as it corresponds to a number rather than to a proposition.

We shall take the formulae 1-5 above as the axioms of our formalism. Having taken the axioms as the initial true (or deducible) formulae, we obtain the remaining deducible formulae by means of the deduction rules of our formalism, i.e. certain constructive operations.

We introduce two rules of deduction.

I. If $A(a)$ is a deducible equation containing the letter a and if β is an arbitrary term, then the equation $A(\beta)$, being the result of replacing the letter a everywhere in A by the term β, is also a deducible equation.

The operation of replacement is given by the operation $R(A, a, B)$ considered above which we called the substitution operation. This, as we know, is a constructive operation.

II. If $A(\alpha)$ is a deducible equation, α a term contained in it, and $\alpha = \beta$ a deducible equation, then the equation $A(\beta)$ obtained by replacing the term α in $A(\alpha)$ by the term β is also a deducible equation.

The operation of replacement of one term by another term is also constructive. Its meaning and finitary-realizable character are entirely obvious.

By the introduction of axioms and rules of deduction, we have defined a constructive class of deducible equations.

The formal treatment of a system of numerical axioms shows that one can abstract from the content of these axioms, considering them simply as rows of elements, and the rules of logical inference as operations on these rows. If we cannot compare the formalism obtained with anything lying outside it, then it will represent in an intrinsic manner the system defined, consisting of the set of rows, which are, for some reason, called equations. Certain rows, which are called deducible equations, are picked out from them. This means only that they are obtained from some chosen rows with the aid of well-defined operations.

The question of consistency cannot be formulated for this formalism since it does not contain negation. But we can formulate another problem for it which has a known analogy with the problem of consistency. We shall call a formalism "void" if every equation is deducible in it. (The analogy of this concept with the concept of consistency is that, as we shall see in the sequel, in every inconsistent system, containing the usual logical principles, all formulae are deducible.) The non-vacuousness of our system is proved very simply, since the formula $a = b$ cannot be deduced in it. This is easily proved utilizing the informal meaning of the formalism. In fact, if the row $a = b$ were true in the formal sense, then the informal numerical equation

$a = b$ would be true for arbitrary numbers a and b, which we know is not the case. That such a proof is possible is due to the unusual formulation of the problem of the non-vacuousness of the formalism inasmuch as in the definition of the rows we are considering and in the operations on them the concept of number does not occur anywhere. Speaking more precisely, the proof is deficient since it depends on the hypothesis of the consistency of the numerical system which is used for the interpretation. However, this interpretation can be made so precise that the problem of consistency for it disappears.

We pose yet one question more concerning our formalism: Do there exist in it equations such that if any one of them is adjoined to the axiom system of the formalism as a new axiom then we again obtain a non-vacuous system? Or, conversely, do we obtain a vacuous system no matter what non-deducible equation we adjoin to the system of axioms? We shall frequently encounter analogous problems in the sequel. If we adjoin to the system of axioms 1-5 the non-deducible formula $a = b$, then the system becomes vacuous. In fact, replacing a and b by arbitrary terms—on the basis of the substitution rule—we shall show that every equation $\alpha = \beta$ is deducible in the system obtained. However, if we adjoin to the system of axioms 1-5 the non-deducible formula

$$ab + c = (a + c)(b + c),$$

we obtain a non-vacuous system. The non-deducibility of the latter formula in our system follows from the fact that it is a false numerical equation. However, one can find another interpretation for the new system of axioms. We shall consider the variables $a, b, c, \ldots$ as finite sets (the empty set included), the expression $(\alpha + \beta)$ as the set consisting of all the elements in the set α and of all the elements in the set β, or, as we say, the set-theoretic sum, and the expression $(\alpha\beta)$ as the set of elements belonging to both α and β, or, the set-theoretic intersection. By the equality $\alpha = \beta$ of the terms α, β we shall understand the coincidence of the sets α and β. Then all axioms, including the new one, are satisfied and the deduction rules lead, as before, only to true identities. But the identity $a = b$ is not true in this formalism either as a and b can be distinct. Thus, the new system is also non-vacuous.

The question of the non-vacuousness of our formalism (analogous to the question of consistency) is easily solved by the method of interpretation. However, we are now no longer concerned with the method of interpretation which, as we indicated above, has its limitations.

We shall consider, for instance, the question of the independence of the first axiom in the formalism under consideration: $a = a$. This question may be formulated in the following way: is the equation $a = a$ deducible from the other axioms by means of the deduction rules or not? If it turns out that it is deducible, then it is superfluous in the system of axioms 1-5 in the sense that the class of deducible equations of the formalism does not change if we

eliminate it. To prove the independence of this axiom by the method of interpretation is still possible but much more difficult. It is significantly simpler to prove its independence by other means. We note that in all remaining axioms the terms connected by the equality sign are configurations which never reduce to a single element. In other words, none of these terms consists of one letter. If we apply our deduction rules to an arbitrary equation possessing this property, we obtain an equation which also possesses this property. In fact, applying the first rule, we replace the letters in an equation by terms and, consequently, we can only complicate the term considered. Applying the second rule, we can never obtain equations in which any part contains only one letter as the term β by which the term α is replaced in the formula $A(\alpha)$ is present in the equation $\alpha = \beta$ which is already introduced by assumption and therefore it itself contains more than one letter. From this it follows that all formulae which are deducible from the system of axioms 2-5 always have the form $\alpha = \beta$, where α and β contain more than one letter. Therefore, the axiom $a = a$ cannot be deduced from the remaining axioms of the formalism.

In the example under consideration, we are dealing with a very weak formalism, but we can, in an analogous way, construct strong systems comprising such deductive means as mathematics makes use of in its various branches: in arithmetic, analysis, theory of functions, and so on.

Hilbert's original idea consisted in reducing all informal mathematical knowledge to finitism and in considering the corresponding mathematical disciplines as the formalisms described above, treating these formalisms as meaningless, being themselves only objects of mathematics. The problems of the foundations of mathematics could then be formulated in terms of the concepts of finitism and one could hope to solve them by the methods of finitism. It is in this direction that the question of the possibility of using the theory of sets should be decided. Expressing the theory of sets by means of formal systems and investigating the problem of the consistency of these systems, we should be able to explain the bounds of application of the set-theoretic conception, or, at least, have indicated bounds in which a contradiction certainly does not arise. Thus, we could introduce the use of actual infinity and we would know when this was possible. At first glance it seems that there are no impediments to the fulfilment of such a programme, but it subsequently came to light that this programme could not be fulfilled in the literal sense of its formulation. Although all mathematical expressions and every logical deduction can indeed be represented in terms of Hilbert's formal systems and in this sense the formalisms can, without limit, comprise all mathematical knowledge, Hilbert's finitism is insufficient for the solution of problems concerning the consistency of fundamental mathematical disciplines. The fact is that the concepts and principles of all mathematics cannot be completely expressed by any formal system no matter how powerful

it may be. This circumstance, in particular, manifests itself in that, as was shown by Gödel, the problem of the consistency of a formal system cannot be solved by methods which are formalizable in this system itself. Since the methods of reasoning allowed by finitism can be expressed within the bounds of a well-defined formalism (for example, in axiomatic arithmetic, which is discussed in Chapter V), the consistency of such a formalism cannot be proved within the framework of finitism. There is, however, no reason to assume that the bounds which are enforced by Hilbert's finitism are in reality necessary in order to exclude those elements of mathematical thinking which raise doubts. Further analysis of the object of mathematics and the isolation in it of safe consistent methods are possible, methods which go beyond the bounds of finitism and are none the less sufficiently powerful in order to solve the problems of interest to us. But passing beyond the bounds of finitism does not destroy the basic idea of the method proposed by Hilbert which consists in the formalization of the mathematical systems under consideration, placing them on a firm foundation by means of some set of concepts which are assumed as fundamental. In fact, though the methods of finitism are insufficient for the solution of the problems indicated above, these methods are entirely sufficient for the formulation of these problems.

One could conclude from what we stated above that, as far as the consistency of certain formalisms is concerned, the only means of judging consistency we have is by their content; in other words, the solution of the consistency problem again requires the method of interpretation. But the content of formalisms, described by set-theoretical systems, as we have already stated above many times, is itself in need of a foundation. None the less, the set-theoretic interpretation—for lack of anything better—is also applied to the study of formalisms. Looking at them from the set-theoretical point of view is called "informal" consideration, although here the term "naïve-informal" consideration is more appropriate. A satisfactory solution of the problems of the foundations of mathematics cannot be given by such a method, and we find ourselves faced with an essential difficulty. However, the content of a formalism is not always obliged to be set-theoretic. A critical contemplation of the foundations of the theory of sets brought to light other—non-set-theoretical representations—which are capable of comprising the content of formalisms free from those elements of set-theoretic conceptions which induce doubt.

There is reason to hope that formalisms whose consistency we judge on the basis of the contents expressed by them, form a set which, although it indeed does not contain all consistent formalisms, is such that the consistency of any formalism can be reduced to the consistency of formalisms of the given set by means of Hilbert's finitism.

As often happens, the new ideas which we have described, which arose from the problems of the foundations of mathematics, developed beyond the

original domain of their problems. They brought in basically new concepts and methods which were applied also to problems not directly connected with the foundations of mathematics. The apparatus of mathematical logic found applications in computational mathematics and in technology in connection with the construction of complicated automatic systems.

CHAPTER I

PROPOSITIONAL ALGEBRA

§1. Logical operations

The study of propositions—called propositional algebra—is the first of formal logical theories. It does not belong to calculi of the type mentioned in the Introduction. But, although these calculi are the fundamental object of our book, we shall begin our discussion of the foundations of mathematical logic with propositional algebra. The fact is that acquaintance with the laws of propositional algebra greatly facilitates the study of those logical calculi which we shall encounter in the sequel. Moreover, propositional algebra is of interest in its own right and it has applications in other branches of science. It is applied, for example, in the synthesis of relay and electronic networks.

We shall consider propositions which we shall assume satisfy the law of the excluded middle and the law of dichotomy, i.e. that every proposition is either true or false and cannot be simultaneously true and false. (There is no need whatsoever for considering these laws of logic to be universally valid. But we shall limit ourselves to the consideration of only those propositions for which these laws hold.) We abstract from the content of a proposition and even from its structure; in particular, we shall not point out its subject and predicate. We shall limit ourselves to its property of being true or false. Then a proposition may be considered as a quantity which can assume one of two values: "truth" and "falsity".

EXAMPLE. Let the following propositions be given: "a dog is an animal"; "Paris is the capital of Italy"; "$3 < 5$"; "in every triangle, the bisector of an angle divides the opposite side into equal parts".

From our point of view, the first of these propositions can be replaced by the symbol "truth", the second by "falsity", the third by "truth", and the fourth by "falsity". Such a treatment of the study of propositions constitutes the subject-matter of propositional algebra. We shall denote propositions by upper case Latin letters A, B, ..., and their values, i.e. their truth or falsity, by T and F, respectively. In this entire chapter, we consider propositions only as quantities which take on the values T or F. In ordinary speech, connectives are required between propositions, such as *and*, *or*, and others, these connectives enabling one to form new propositions by combining

various propositions with one another. For example, take the connective "*and*". Let "π is greater than 3" and "π is less than 4" be given propositions; we can form a new proposition "π is greater than 3 and π is less than 4". The proposition "if π is irrational, then π^2 is also irrational" is obtained by combining two propositions by means of the connective "*if—then*". Finally, we can obtain a new proposition from a given proposition by negating it. Considering propositions as quantities which assume the values T and F, we define operations on them which enable us to obtain new ones from given propositions. These operations essentially express the connectives, mentioned above, which are utilized in ordinary speech.

Let A and B be two arbitrary prescribed propositions.

1. The first operation on these propositions consists in forming a new proposition which will be denoted by $A \mathbin{\&} B$ and which is true if, and only if, A and B are true. In ordinary speech, the combination of propositions by means of the connective "*and*" corresponds to this operation.

2. The second operation on the propositions A and B, which is expressed in the form $A \vee B$, is defined as follows: it is true if, and only if, at least one of the initial propositions is true.

In ordinary speech, this operation corresponds to the combination of propositions by means of the connective "*or*". However, here we do not have the exclusive "*or*" which is understood in the sense of "*either—or*" when A and B cannot both be true. In our definition, the proposition $A \vee B$ is also true when both the propositions A and B are true.

3. The third operation is denoted by $A \rightarrow B$; this proposition is false if, and only if, A is true and B is false. A is called the *antecedent* and B the *consequent*; the operation $A \rightarrow B$ is called *implication*. In ordinary speech, this operation corresponds to the connective "*if—then*": "if A, then B". But, in our definition, this proposition is always true for false A independently of whether or not the proposition B is false. This situation can be formulated briefly as follows: "a false proposition implies anything we please". It is sometimes understood, in ordinary speech, that when A is false then the proposition "if A, then B" is devoid of meaning. However, here we cannot take such an interpretation. In fact, let us assume, for example, that some reduction has been proved—which reduces the hypothesis B, which is some assertion in number theory, to the Riemann hypothesis, which we denote by A. It is not known whether "A implies B" is true. Thus, we assume that the assertion "A implies B" is true in this case, although A can be false. On the other hand, this reduction is of interest only when it is unknown whether the antecedent A is true. In fact, if we knew that the antecedent were true, then the reduction would amount to the proof of B.

Moreover, the concept of implication used in ordinary speech has still another shade of meaning. The proposition "it follows from the fact that a lion has claws that the snow is white" is true in the sense we have defined.

In fact, the proposition "the snow is white", appearing here as the consequent, is true, and therefore the entire assertion is true independently of the truth or falsity of the antecedent. But it does not follow at all from the fact that a lion has claws that the snow is white with the usual interpretation of the concept of implication since it is understood that the consequent must somehow be deduced from the antecedent. But this cannot be done if the content of the antecedent and consequent are absolutely unrelated. Such an interpretation of implication cannot be in any way defined in the logical calculus being considered inasmuch as it cannot be formulated in terms of truth and falsity only.

4. $\bar{A}$ is a false proposition when A is true and true when A is false. The proposition $\bar{A}$ is called the ***negation*** of A.

5. $A \sim B$ is a true proposition if, and only if, A and B are both true or both false. This proposition is called *equivalence*.

Let X, Y, Z, U, V, W, ... be arbitrary propositions, i.e. from our point of view, they are quantities which take on one of the two values T and F.

With the aid of the operations &, $\vee$, $\to$, $\sim$ and —, we can form the following compound propositions from them:

(1) $X \,\&\, Y$, (2) $X \vee Y$, (3) $X \to Y$, (4) $\bar{X}$, (5) $X \sim Y$. From the stock of propositions just obtained we can, using these same operations, obtain new compound propositions—for example:

$$X \to (Y \vee Z), \quad \overline{X \sim Y}, \quad \overline{(X \to Y)} \to (X \sim \overline{(U \,\&\, V)}),$$
$$X \vee (Y \,\&\, Z),$$
$$\overline{\overline{(X \vee Y)} \,\&\, (Z \to (U \to (V \sim W)))}, \quad X \,\&\, (Y \,\&\, (Z \,\&\, (U \,\&\, (V \,\&\, W)))),$$

and so on. Knowing the values that the propositions X, Y, ..., W take, we can easily establish the value of a compound proposition constructed from them. For example:

1. Suppose X is T, Y is F, Z is F; then the compound proposition $X \to (Y \vee Z)$ can be written in the form: $T \to (F \vee F)$. The value of this proposition is F. In fact, $F \vee F$ is F; $T \to F$ is also F.

2. Suppose X is F, Y is F, Z is T, U is T, V is F and W is F. Consider the compound proposition

$$\overline{\overline{(X \vee Y)} \,\&\, (Z \to (U \to (V \sim W)))}.$$

It can be written in the form

$$\overline{\overline{(F \vee F)} \,\&\, (T \to (T \to (F \sim F)))}.$$

In this case, $F \vee F$ is F and $\overline{F \vee F}$ is T. Therefore, the compound proposition can be rewritten as

$$\overline{T \,\&\, (T \to (T \to (F \sim F)))}.$$

But $F \sim F$ is T and $T \to T$ is also T; therefore, the proposition appearing

under the negation sign has the value T. Then, the entire proposition has the form $\bar{T}$, i.e. it is F.

Every compound proposition constructed from certain initial propositions using the logical operations 1-5 *will be called a formula of propositional algebra.*

In this connection, the initial propositions can be constant, i.e. they can have a definite value T or F, or they may not have a definite value. In the latter case they are denoted by upper case Latin letters. In the former case, we shall call the initial propositions *constant elementary propositions* and in the latter case *variable elementary propositions.* If we assign values to all variable elementary propositions, then the formula itself will assume a definite value. Thus, every formula defines some function whose arguments are variable elementary propositions.

In the sequel, we shall be dealing primarily with formulae which contain only variable elementary propositions. We shall assume that if no special assumptions are made about a formula then it contains only variable elementary propositions. Since the arguments and functions are capable of assuming only two distinct values, such a function can be completely described by a finite table. The following are the tables for the simplest functions $X \,\&\, Y$, $X \vee Y$, $X \rightarrow Y$, $X \sim Y$, $\bar{X}$, respectively:

X	Y	$X \,\&\, Y$
T	T	T
F	T	F
T	F	F
F	F	F

X	Y	$X \vee Y$
T	T	T
F	T	T
T	F	T
F	F	F

X	Y	$X \rightarrow Y$
T	T	T
F	T	T
T	F	F
F	F	T

X	Y	$X \sim Y$
T	T	T
F	T	F
T	F	F
F	F	T

X	$\bar{X}$
T	F
F	T

.

§2. Equivalence of formulae

Propositions, which can be as complicated as we please, can be formed by means of the operations on propositions, introduced above. For example,

$$(A \,\&\, B) \vee C; \quad ((A \rightarrow B) \sim C) \,\&\, \overline{((A \vee \bar{B}) \,\&\, C)}.$$

Every formula is a function of the letters A, B, ... which appear in it. *Two formulae* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are said to be equivalent if they take on the same value for any values of* $X_1, X_2, \ldots, X_n$, *where* $X_1, X_2, \ldots, X_n$ *is the set of variables occurring in* $\boldsymbol{A}$ *and* $\boldsymbol{B}$.

EXAMPLES:

$$\bar{\bar{X}} \text{ is equivalent to } X;$$
$$X \vee X \text{ is equivalent to } X;$$
$$(X \,\&\, \bar{X}) \vee Y \text{ is equivalent to } Y;$$
$$X \vee \bar{X} \text{ is equivalent to } Y \vee \bar{Y}.$$

The following connection exists between the concept of equivalence and the equivalence symbol $\sim$: *if the formulae* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are equivalent, then the formula* $\boldsymbol{A} \sim \boldsymbol{B}$ *assumes the value T for all values of the variables*, and conversely: *if the formula* $\boldsymbol{A} \sim \boldsymbol{B}$ *assumes the value T for all value of the variables, then the formulae* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are equivalent.*

The validity of this assertion follows directly from the definition of the operation $\sim$. It is easily seen that the relation of equivalence of two formulae is symmetric and transitive.

In the definition of the equivalence of two formulae, it is not necessary to assume that they contain the same variables. Thus, in the third and fourth examples above, we have the case when different variables occur in equivalent formulae. It is, moreover, clear that if a variable occurs in only one of two equivalent formulae, then this formula takes on the same value for all values of this variable if the values of the other variables remain fixed. In other words, although this variable does indeed appear in the formula, the function defined by the formula does not depend on this variable.

The following are the most important examples of equivalent formulae:

$$\bar{\bar{X}} \text{ is equivalent to } X. \tag{1}$$
$$X \,\&\, Y \text{ is equivalent to } Y \,\&\, X. \tag{2}$$
$$(X \,\&\, Y) \,\&\, Z \text{ is equivalent to } X \,\&\, (Y \,\&\, Z). \tag{3}$$
$$X \vee Y \text{ is equivalent to } Y \vee X. \tag{4}$$
$$(X \vee Y) \vee Z \text{ is equivalent to } X \vee (Y \vee Z). \tag{5}$$
$$X \,\&\, (Y \vee Z) \text{ is equivalent to } (X \,\&\, Y) \vee (X \,\&\, Z). \tag{6}$$
$$X \vee (Y \,\&\, Z) \text{ is equivalent to } (X \vee Y) \,\&\, (X \vee Z). \tag{7}$$
$$\overline{(X \vee Y)} \text{ is equivalent to } \bar{X} \,\&\, \bar{Y}. \tag{8}$$

$$\overline{(X \,\&\, Y)} \text{ is equivalent to } \bar{X} \vee \bar{Y}. \tag{9}$$

$$X \vee X \text{ is equivalent to } X. \tag{10}$$

$$X \,\&\, X \text{ is equivalent to } X. \tag{11}$$

$$X \,\&\, T \text{ is equivalent to } X. \tag{12}$$

$$X \vee F \text{ is equivalent to } X. \tag{13}$$

Relations (1)-(13) are easily verified on the basis of the definitions of the operations &, V and —. Since equivalent formulae are identical, i.e. they can be substituted for one another—the equivalence relations allow us to perform transformations on formulae, thereby reducing them to a simpler or to a more convenient form.

For example, $((X \vee Y) \,\&\, Y) \vee (X \vee X)$ is equivalent to $(X \,\&\, Y) \vee (X \vee X)$, which in turn is equivalent to $(X \,\&\, Y) \vee X$.

In the same way, in an arbitrary formula, one can replace any part of it by an equivalent formula and the formula obtained is equivalent to the initial formula.

Relations (2), (3), (4), (5) show that operations defined by the symbols & and V are subject to the commutative and associative laws. Therefore, if a formula A is constructed from the formulae $A_1, A_2, \ldots, A_n$ using the operation & only, then we always obtain a formula which is equivalent to the formula A no matter in what order we perform these operations. We shall write such a formula A in the form

$$A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n,$$

in which all brackets round the formulae A_i are omitted. In exactly the same way, if the formula A is constructed from the formulae $A_1, A_2, \ldots, A_n$ using only the operation V, then we shall write it in the form

$$A_1 \vee A_2 \vee \ldots \vee A_n.$$

Relation (6) shows that the operation & is distributive with respect to the operation V, in the same way that in ordinary arithmetic multiplication is distributive with respect to addition. In virtue of this analogy, we shall call the operation & multiplication and the operation V addition.

The expression $A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n$ will be called a product and its terms A_i will be called factors. The symbol & is sometimes omitted.

The expression $A_1 \vee A_2 \vee \ldots \vee A_n$ will be called a sum and its terms A_i will be called summands.

The analogy between the commutative and associative laws of addition and the law of distribution of multiplication with respect to addition in propositional algebra and the same laws for the addition and multiplication of numbers leads to the possibility of performing the transformations of removing brackets, inserting brackets and moving a common factor outside brackets as one does in ordinary algebra.

In virtue of relation (7), the operation $\vee$ is also distributive with respect to the operation &. Therefore, conversely, one sometimes calls the operation $\vee$ multiplication and the operation & addition and in this way the analogy with algebra, indicated above, is retained.

Transformations which represent the application of the distributive laws (6) and (7) will be called *distributive operations. Relations* (6) *and* (7) *will be called the first and second distributive laws, respectively.*

One can simplify the writing of formulae still further by omitting certain brackets and by assuming, in this connection, that the multiplication operation has preference over the addition operation and that both the addition and multiplication operations have priority over the operations $\rightarrow$ and $\sim$. Or, as we also say: multiplication *connects more strongly* than addition and both multiplication and addition *connect more strongly* than the operations $\rightarrow$ and $\sim$. Furthermore, we shall assume that the symbol —, occurring over a formula, makes the brackets in which this formula is enclosed superfluous. Thus, for example, the formula

$$XY \vee ZU$$

is understood to be

$$(X \,\&\, Y) \vee (Z \,\&\, U),$$

the formula

$$X \vee Y \rightarrow ZU$$

as

$$(X \vee Y) \rightarrow (Z \,\&\, U),$$

and the formula

$$\overline{X \vee Y \,\&\, Z}$$

as

$$(\overline{X \vee Y) \,\&\, Z}.$$

A system of elements for which operations of addition, multiplication and negation are defined and which satisfy the relations (1)-(13) is called a *Boolean algebra.* Thus, one can say that propositions and the fundamental logical operations &, $\vee$, — form a Boolean algebra.

There exist, however, other collections of objects (i.e. not logical systems) which also form a Boolean algebra. For example, the collection of subsets of a set R for which addition is set-theoretic addition, multiplication is set-theoretic multiplication, and negation is taking the complement relative to R, is also a Boolean algebra (in which the entire set R plays the role of T and the empty subset plays the role of F).

We now exhibit yet another example of a Boolean algebra.

Let M be any bounded set of real numbers which contains its least upper bound p and greatest lower bound q. Suppose, furthermore, that M is symmetric with respect to the point $(p + q)/2$, which will be called the centre of the set M. In other words, if $x \in M$, then the point x' which is situated symmetrically to x with respect to the centre also belongs to M. The opera-

tions of addition, multiplication and negation are defined as follows. Retaining the symbols of logic for these operations, we set

$$x \vee y = \max(x, y), \quad x \,\&\, y = \min(x, y),$$

and $\bar{x}$ is the point which is symmetric to x with respect to the centre of the set M. It is easily verified that all relations (1)-(13) are satisfied for the indicated operations (here p plays the role of T and q that of F).

In the case when the set M consists of two numbers 0 and 1, this system represents a propositional algebra in which the symbols F and T are replaced by the numbers 0 and 1, respectively.

The logical operations &, $\vee$, $\rightarrow$, $\sim$ and — are not mutually independent. Certain of them can be expressed in terms of others in such a way that equivalent formulae are obtained. For example, the symbol $\sim$ can be expressed in terms of the symbols $\rightarrow$ and & in virtue of the relation

$$X \sim Y \text{ is equivalent to } (X \rightarrow Y) \,\&\, (Y \rightarrow X), \tag{14}$$

which fact is easily proved on the basis of the definitions of the operations $\sim$, $\rightarrow$ and &.

The symbol $\rightarrow$ can be expressed in terms of the symbols $\vee$ and — :

$$X \rightarrow Y \text{ is equivalent to } \bar{X} \vee Y.$$

Thus, the symbol $\sim$ can be expressed in terms of the symbols &, $\vee$ and — :

$$X \sim Y \text{ is equivalent to } (\bar{X} \vee Y) \,\&\, (\bar{Y} \vee X). \tag{15}$$

The symbol $\sim$ can be expressed in terms of the symbols &, $\vee$ and — in yet another way:

$$X \sim Y \text{ is equivalent to } X \,\&\, Y \vee \bar{X} \,\&\, \bar{Y}, \tag{16}$$

which can also easily be proved.

Thus, the symbols $\rightarrow$ and $\sim$ can be expressed in terms of the symbols &, $\vee$ and —. One can go even further and exclude still one more symbol, & or $\vee$, whichever we wish.

We shall show how to express & in terms of $\vee$ and —.

In virtue of relation (8), we have that

$$\bar{\bar{X}} \,\&\, \bar{\bar{Y}} \text{ is equivalent to } \overline{\bar{X} \vee \bar{Y}}.$$

On the basis of (1), $\bar{\bar{X}}$ and $\bar{\bar{Y}}$ can be replaced by X and Y, respectively, and, consequently,

$$X \,\&\, Y \text{ is equivalent to } \overline{\bar{X} \vee \bar{Y}}.$$

The symbol & is thus expressed in terms of the symbols $\vee$ and —.

Thus, all operations can be replaced by two: $\vee$ and — by means of equivalent expressions.

In an analogous manner, using (9), we can replace all operations by & and —.

REMARK. If the formula A contains only the operations &, $\vee$ and —,

then, in virtue of relations (1), (8) and (9), it can be transformed into a form in which the negation symbols will operate only on elementary propositions. In fact, if the negation symbol occurs over a sum: $\overline{A \vee B}$, then this formula can be replaced, on the basis of (9), by the product $\bar{A} \,\&\, \bar{B}$; if the symbol — occurs over a product: $\overline{A \,\&\, B}$, then this formula can be replaced by the sum $\bar{A} \vee \bar{B}$; and if the negation symbol occurs over the negation symbol: $\bar{\bar{A}}$, then, in virtue of (1), both these symbols can be removed. Performing such transformations, we reduce the formula to a form in which the negation symbol is applied only to elementary propositions.

EXAMPLES:

1. $\overline{\bar{X} \vee \bar{Y}}$.

Transforming this formula, using relation (8), we obtain the following equivalent formula:

$$\bar{\bar{X}} \,\&\, \bar{\bar{Y}}.$$

In virtue of relation (1), the double negation symbols can be removed, and then we finally obtain the formula

$$X \,\&\, Y.$$

2. $\overline{X \,\&\, Y \vee \bar{Z}}$.

Transforming this formula, using relation (8), we obtain the formula

$$\overline{X \,\&\, Y} \,\&\, \bar{\bar{Z}}.$$

Transforming the first factor, using relation (9), we obtain

$$(\bar{X} \vee \bar{Y}) \,\&\, \bar{\bar{Z}}.$$

Deleting the double negation in the second factor, we finally obtain

$$(\bar{X} \vee \bar{Y}) \,\&\, Z.$$

§3. Duality law

In this section we shall consider formulae which contain only the operations &, V and —. As was already established above, every formula can be reduced to this form by means of equivalence transformations.

We shall say that *the operation & is dual to the operation* V *and conversely.* We shall also introduce the concept of dual formulae. *The formulae A and A^* are said to be dual if either one can be obtained from the other by replacing every operation by its dual.*

EXAMPLES:

1. $(X \vee \bar{Y}) \,\&\, Z$ — $X \,\&\, \bar{Y} \vee Z$,
2. $\overline{X \vee Y} \,\&\, (X \vee \overline{Y \,\&\, Z})$ — $\overline{X \,\&\, Y} \vee X \,\&\, \overline{Y \vee Z}$,
3. $X \,\&\, Y \vee Y \,\&\, Z \vee U \,\&\, V$ — $(X \vee Y) \,\&\, (Y \vee Z) \,\&\, (U \vee V)$,
4. $X \,\&\, (Y \vee Z \,\&\, (U \vee V))$ — $X \vee Y \,\&\, (Z \vee U \,\&\, V)$.

The duality relation is mutual for formulae as well as for operations: if A^* is dual to A, then also, conversely, A is dual to A^*.

One can easily deduce the following result from equivalences (8) and (9): *if $A(X_1, \ldots, X_n)$ and $A^*(X_1, \ldots, X_n)$ are dual formulae and $X_1, \ldots, X_n$ are all the elementary propositions occurring in them, then $\bar{A}(X_1, \ldots, X_n)$ is equivalent to $A^*(\bar{X}_1, \ldots, \bar{X}_n)$.*

From this relation in turn there follows the so-called *duality law* which is formulated as follows.

If the formulae A and B are equivalent, then the formulae A^* and B^* which are dual to them are also equivalent.

Let $A(X_1, \ldots, X_n)$ and $B(X_1, \ldots, X_n)$ be equivalent formulae, and let $X_1, \ldots, X_n$ be the elementary propositions occurring in them. Then

$$A^*(X_1, \ldots, X_n) \text{ is equivalent to } \bar{A}(\bar{X}_1, \ldots, \bar{X}_n)$$

and

$$B^*(X_1, \ldots, X_n) \text{ is equivalent to } \bar{B}(\bar{X}_1, \ldots, \bar{X}_n).$$

The equivalence of the formulae $A(X_1, \ldots, X_n)$ and $B(X_1, \ldots, X_n)$ implies the equivalence of the formulae $A(\bar{X}_1, \ldots, \bar{X}_n)$ and $B(\bar{X}_1, \ldots, \bar{X}_n)$, since, in virtue of the definition of equivalence, $A(X_1, \ldots, X_n)$ and $B(X_1, \ldots, X_n)$ have the same values for arbitrary values of the variables $X_1, \ldots, X_n$ and, consequently, for the values $\bar{X}_1, \ldots, \bar{X}_n$ also.

In virtue of what we have just stated, the formulae $A(\bar{X}_1, \ldots, \bar{X}_n)$ and $B(\bar{X}_1, \ldots, \bar{X}_n)$ are equivalent—and so the formulae $\bar{A}(\bar{X}_1, \ldots, \bar{X}_n)$ and $\bar{B}(\bar{X}_1, \ldots, \bar{X}_n)$ are also equivalent. Since $A^*(X_1, \ldots, X_n)$ and $B^*(X_1, \ldots, X_n)$ are equivalent to the formulae $\bar{A}(\bar{X}_1, \ldots, \bar{X}_n)$ and $\bar{B}(\bar{X}_1, \ldots, \bar{X}_n)$, respectively, they are mutually equivalent.

If, in applying the distributive laws to the formula A, we obtain, on the basis of the first distributive law, the formula B, then the transition from the dual formula A^* to the dual formula B is realized by means of distributive transformations on the basis of the second distributive law. The transition from A^* to B^* will be called the dual transformation to that transformation carrying A into B.

§4. Decision problem

We shall say that a formula is *identically true* if it has the value T for all values of the variable propositions occurring in it. The following are examples of identically true formulae:

$$1.\ X \vee \bar{X}, \qquad 2.\ X \to (Y \to X), \qquad 3.\ (X \,\&\, (X \to Y)) \to Y.$$

We shall say that a formula is *satisfiable* if it has the value T for some values of the variable propositions occurring in it. The formulae

$$1.\ X, \qquad 2.\ X \vee \bar{Y}, \qquad 3.\ X \to \bar{X}$$

are satisfiable.

We shall say that a formula is *not satisfiable* or *identically false* if it has the value F for all values of the variables occurring in it. The negation of an identically true formula will obviously be an identically false formula, and conversely.

We can formulate the following problem: give a procedure which will enable one to determine in a finite number of operations whether a prescribed formula is identically true or not. Having such a procedure, we have, by the same token, a method for recognizing whether or not a given formula is satisfiable. In fact, if it is possible in a finite number of operations to verify whether or not any formula is identically true or not, we can, for an arbitrary formula A, solve the problem whether $\bar{A}$ is an identically true formula or not. If $\bar{A}$ turns out to be identically true, then this means that A is identically false, and, consequently, it is not satisfiable; if $\bar{A}$ is not identically true, then A is not identically false and, hence, it is satisfiable.

The problem just formulated is called the "*decision problem*" and it is posed not only for propositional algebra but also for other logical systems. For propositional algebra, this problem is easily solved.

Let $A(X_1, \ldots, X_n)$ be a formula, in the propositional calculus, containing the elementary propositions $X_1, \ldots, X_n$. This formula defines some function of the variables $X_1, \ldots, X_n$, where the variables $X_1, \ldots, X_n$ as well as the function A can have only two values; the number of possible combinations of the values of the variables $X_1, \ldots, X_n$ is finite and is precisely equal to 2^n. For each such combination, we can determine the value of the formula A by replacing $X_1, \ldots, X_n$ by their values and then computing the value of the formula A which, as we know, is arrived at after a finite number of operations. Once we know the value of the formula A for each combination of values of the variables $X_1, \ldots, X_n$, we can decide whether or not it is identically true.

The procedure just discussed of course yields the basic solution of the decision problem, but the number of trials which must be made even for uncomplicated formulae is so large that such a direct verification is not feasible in practice.

There exists another procedure based on the reduction of formulae to the so-called "normal form". This normal form is also used in other problems of mathematical logic.

A product (sum) of variables and their negations will be called an *elementary product* (*elementary sum*). The expression "sum of elementary products" ("product of elementary sums") will be understood to be extended so as to include the case when the sum reduces to one term (or the product reduces to one factor).

THEOREM 1. *A necessary and sufficient condition for an elementary sum to be identically true is that it contain at least one pair of terms of which one is a variable and the other is its negation.*

Proof. The condition is *sufficient*. In fact, if such a pair of terms can be

found, then the sum has the form

$$X \vee \bar{X} \vee Y \vee Z \ldots$$

(where the terms Y, Z, ... can be absent). But the sum $X \vee \bar{X}$ is identically true, and therefore the entire sum under consideration is identically true no matter what the terms Y, Z, ... are.

The condition is *necessary*. Let us assume that the sum does not contain a pair of terms of which one is the negation of the other. In this case we can assign the value F to every variable which does not occur under the negation symbol and the value T to every variable which does occur under the negation symbol. This can be done inasmuch as none of the variables occurs in the sum simultaneously with negation and without negation. After the substitution indicated, every term has the value F. Then the entire elementary sum has the value F also and, consequently, the formula is not identically true, which is what we were required to prove.

The following theorem is proved analogously.

THEOREM 2. *A necessary and sufficient condition for an elementary product to be identically false is that it contain at least one pair of factors of which one is the negation of the other.*

A formula which is equivalent to a given formula and is the sum of elementary products is called the disjunctive normal form of the given formula.

As we have already stated, all logical operations can be reduced to three: &, $\vee$ and —. We shall assume that the formulae whose normal forms we shall determine contain only these operations. The symbol — can be assumed to be applied only to elementary propositions (see the remark at the end of §2).

As we saw above, a formula constructed of variables and their negations can be subjected, with the aid of the operations & and $\vee$, to the same transformations as can algebraic expressions. Consequently, one can remove all brackets and write every formula of this form as the sum of elementary products. We have thus proved that the disjunctive normal form exists for any formula.

EXAMPLE: We shall find the disjunctive normal form of the formula

$$X \,\&\, (\overline{\bar{Y} \,\&\, Z}) \,\&\, (U \vee V).$$

We first reduce this formula to the form in which negation applies only to elementary propositions:

$$X \,\&\, (Y \vee \bar{Z}) \,\&\, (U \vee V).$$

Then, applying the distributive law, we remove the brackets by performing operations analogous to what we do in the multiplication of polynomials. We then obtain

$$(XY \vee X\bar{Z})\,(U \vee V)$$

and further

$$XYU \vee XYV \vee XZU \vee XZV.$$

The formula just obtained is the disjunctive normal form of the initial formula.

The conjunctive normal form of a given formula is a formula which is equivalent to it and is the product of elementary sums.

We shall prove that the conjunctive normal form exists for every formula.

First, suppose A is an arbitrary formula which contains only the operations &, $\vee$ and —. We consider the formula A^* which is the dual of A. Let B^* be the disjunctive normal form of the formula A^* and let B be the formula dual to B^*. Since A^* and B^* are equivalent, we have, by the duality law, that A and B are also equivalent. Formula B^*, being in the disjunctive normal form, is the sum of elementary products. It is easily seen that the formula B, which is the dual of B^*, is the product of elementary sums. And since B is equivalent to A, we consequently have that it is the conjunctive normal form of formula A. Since, for every formula, there exists a formula containing only &, $\vee$ and — which is equivalent to it, we consequently have that the conjunctive normal form exists for every formula, which is what we were required to prove.

EXAMPLES: We shall find the disjunctive and conjunctive normal forms for the following formulae.

1. $X(X \to Y)$.

We first of all eliminate the sign $\to$ by replacing the formula by the following:

$$X(\bar{X} \vee Y).$$

This formula is already in the conjunctive normal form. Using the distributive law, we can remove the brackets and obtain the disjunctive normal form:

$$X\bar{X} \vee XY.$$

2. $\overline{X \vee Y} \sim XY$.

We eliminate the sign $\sim$ making use of formula (16), §2. We obtain the formula

$$\overline{X \vee Y}\, XY \vee (X \vee Y)\, \overline{XY}.$$

We transform this formula so that the negation sign occurs over elementary propositions:

$$\bar{X}\bar{Y}XY \vee (X \vee Y)(\bar{X} \vee \bar{Y}).$$

Removing the brackets, by use of the distributive laws, we obtain the disjunctive normal form:

$$\bar{X}\bar{Y}XY \vee X\bar{X} \vee X\bar{Y} \vee Y\bar{X} \vee Y\bar{Y}.$$

To obtain the conjunctive normal form, using formula (15), §2, we replace the initial formula by the product of two sums:

$$((X \vee Y) \vee XY)\,(\overline{XY} \vee \overline{X \vee Y}).$$

We transform this formula into a form in which the negation signs occur only over elementary propositions:

$$(X \vee Y \vee XY)\,(\bar{X} \vee \bar{Y} \vee \bar{X}\bar{Y}).$$

Applying the second distributive law, we obtain the conjunctive normal form of the formula:

$$(X \vee Y \vee X)\,(X \vee Y \vee Y)\,(\bar{X} \vee \bar{Y} \vee \bar{X})\,(\bar{X} \vee \bar{Y} \vee \bar{Y}).$$

We note that for every formula A there exists not just one disjunctive normal form and not just one conjunctive normal form. Performing the distributive operations in various ways, we can arrive at different normal forms. Let us consider, for instance, the formula

$$X \vee Y \,\&\, Z.$$

This formula is in the disjunctive normal form. However, it can also be put into another disjunctive normal form by means of distributive operations. Applying the second distributive law, we obtain

$$(X \vee Y)\,(X \vee Z).$$

Applying the first distributive law to this formula, we obtain that

$$XX \vee XZ \vee YZ \vee YZ.$$

This formula is also a disjunctive normal form of the formula $X \vee YZ$. Of course, different normal forms are different only in form. All of them must be equivalent. In the sequel (see §6), we shall isolate from among the normal forms of a given formula the so-called *principal normal form*—the disjunctive as well as the conjunctive.

Utilizing normal forms, it is possible to indicate a simpler solution of the decision problem than the method of direct verification. We shall assume that formula A is identically true. We consider its conjunctive normal form A'. It has the form of a product $A'_1 \ldots A'_n$, where each factor is an elementary sum (in a particular case, n can be equal to unity). Since A' is identically true, every factor must be an identically true formula. But A'_i is an elementary sum, and, by Theorem 1, such a sum can be identically true if, and only if, it contains some variable together with its negation. Therefore, if formula A is identically true, then every factor in its conjunctive normal form contains, as a term, some variable and its negation. We have thus obtained a criterion for a formula to be identically true:

A necessary and sufficient condition for a formula to be identically true is that every factor in its conjunctive normal form have at least two terms of which one is some variable and the other is its negation.

In virtue of the symmetry of the operations of addition and multiplication, one can show, in the logical calculus, in an analogous manner, that:

A necessary and sufficient condition for a formula to be identically false is that every term in its disjunctive normal form contain at least one pair of factors of which one is some variable and the other is its negation.

The criteria obtained yield a complete solution of the decision problem.

EXAMPLES:

1. Decide whether or not the formula

$$Y \vee X\bar{Y} \vee \bar{X}\bar{Y}$$

is identically true.

Taking $\bar{Y}$ out of the brackets in the last two terms, we obtain

$$Y \vee \bar{Y}(X \vee \bar{X}).$$

Applying the second distributive law, we obtain the conjunctive normal form of our formula:

$$(Y \vee \bar{Y})\,(Y \vee X \vee \bar{X}).$$

Since every factor of this formula contains some variable together with its negation, the formula is identically true.

2. Decide whether or not

$$X \,\&\, Y \,\&\, Z \vee \bar{X} \,\&\, \bar{Y} \,\&\, Z \vee X \,\&\, \bar{Y} \,\&\, \bar{Z}.$$

is an identically true formula.

We shall reduce this formula to conjunctive normal form. Applying the second distributive law, we obtain a series of parentheses connected by the sign &:

$$(X \vee \bar{X} \vee X)\,(X \vee \bar{X} \vee \bar{Y})\,\ldots.$$

The parentheses

$$(X \vee \bar{Y} \vee \bar{Z})$$

is found among the factors of this conjunctive normal form. This term does not contain any variable together with its negation and therefore it is not identically true. Consequently, neither is the initial formula identically true. We shall decide whether or not it is satisfiable. To this end we note that the formula itself is in disjunctive normal form. And since none of its terms contains a variable together with its negation, it is not identically false; consequently, it is satisfiable.

§5. Representation of an arbitrary two-valued function by means of formulae from propositional algebra

Let $\Phi(X_1, \ldots, X_n)$ be an arbitrary function of the n variables $X_1, \ldots, X_n$, where the variables and the function itself take on only the two values T and F. We ask the following question: is it possible to define this function by means of some formula of propositional algebra?

Consider the formula

$$\Phi(T, \ldots, T)X_1 \ldots X_n \vee \Phi(T, \ldots, T, F) \mathbin{\&} X_1 \ldots X_{n-1}\bar{X}_n \vee \ldots$$
$$\ldots \vee \Phi(F, \ldots, F)\bar{X}_1 \ldots \bar{X}_n. \qquad \text{(a)}$$

Every term in this sum is a product in which the first factor is the value of the function Φ for certain well-defined values of the variables $X_1, \ldots, X_n$ and the remaining factors are variables X_i or negations of them. In this connection, those and only those variables whose appearance in the first factor takes the value F appear under the negation sign.

For example,

$$\Phi(T, F, F, \ldots, T, F)X_1\bar{X}_2\bar{X}_3 \ldots X_{n-1}\bar{X}_n.$$

Moreover, the sum under consideration contains all possible terms of the indicated form. It is easily seen that formula (a) defines exactly the function $\Phi(X_1, \ldots, X_n)$.

For instance, let $X_1, X_2, \ldots, X_n$ assume the values $F, T, \ldots, T$, respectively. Then the function takes on the value $\Phi(F, T, \ldots, T)$.

We consider the term

$$\Phi(F, T, \ldots, T)\bar{X}_1X_2 \ldots X_n$$

in formula (a). If the variables $X_1, \ldots, X_n$ in this term are assigned the same values that they have in the first factor, i.e. $F, T, \ldots, T$, respectively, then we obtain the expression

$$\Phi(F, T, \ldots, T)\bar{F}T \ldots T.$$

In this expression, all factors—with the possible exception of the first—have the value T inasmuch as the negation signs occur only over the F symbols whereas the T symbols appear without the negation sign. In this case, on the basis of equality (12) one can omit all these true factors from the product. Consequently, the term under consideration is equivalent to the first factor $\Phi(F, T, \ldots, T)$. In every other term the negation signs over variables are distributed differently than in the term being considered, so that by replacing the variables by the same values in the product, either the F symbol occurs without negation or the T symbol occurs under the negation sign. In this case, one of the factors has the value F and, consequently, the entire product has the value F. So then, in the substitution under consideration of the values T and F for the variables, all the terms, except one, have the value F, and one term has the value $\Phi(F, T, \ldots, T)$ which is the value of the function Φ for prescribed values of the variables. On the basis of equivalence (13), all the terms except $\Phi(F, T, \ldots, T)$ can be omitted from the sum, and the sum itself has the same value as the term $\Phi(F, T, \ldots, T)$. Thus, for an arbitrary substitution for the values for $X_1, \ldots, X_n$ in (a), this formula has the same value as the function Φ for the same substitution.

We have thus proved that *every two-valued function* $\Phi(X_1, \ldots, X_n)$ *can*

be represented in the form of a formula of propositional algebra—namely, in the form of formula (a).

The above representation is also applicable to the function $\bar{\Phi}(X_1, \dots, X_n)$. We have that

$$\bar{\Phi}(X_1, \dots, X_n) \text{ is equivalent to } \bar{\Phi}(T, \dots, T)X_1 \dots X_n \vee \dots$$
$$\dots \vee \bar{\Phi}(F, \dots, F)\bar{X}_1 \dots \bar{X}_n.$$

Going over from these equivalent formulae to their negations, we obtain that

$$\Phi(X_1, \dots, X_n) \text{ is equivalent to}$$
$$\overline{\bar{\Phi}(T, \dots, T)X_1 \dots X_n \vee \dots \vee \bar{\Phi}(F, \dots, F)\bar{X}_1 \dots \bar{X}_n}.$$

Performing transformations on the basis of equivalences (1), (8) and (9), §2, we find that

$$\Phi(X_1, \dots, X_n) \text{ is equivalent to } (\Phi(T, \dots, T) \vee$$
$$\vee \bar{X}_1 \vee \dots \vee \bar{X}_n) \,\&\, \dots \,\&\, (\Phi(F, \dots, F) \vee X_1 \vee \dots \vee X_n).$$

Thus, we have found another form for representing a two-valued function:

$$(\Phi(F, \dots, F) \vee X_1 \vee \dots \vee X_n) \,\&\, \dots$$
$$\dots \,\&\, (\Phi(T, \dots, T) \vee \bar{X}_1 \vee \dots \vee \bar{X}_n). \qquad \text{(b)}$$

It is easy to obtain formulae which contain only variable elementary propositions for the representation of a two-valued function. We shall assume that the function $\bar{\Phi}(X_1, \dots, X_n)$ is not identically false. In this case it takes on the value T for certain values of the variables. In the representation of our function by means of formula (a), only those terms can have the value T for which the first factor has the value T (since the entire product is false whenever one of the factors is false). On the other hand, on the basis of relation (13), §2, false terms can be omitted from the sum. Therefore, in the sum (a), we can leave only those terms in which the first factor has the value T. There must be such terms in the case under consideration. After deleting the false terms in each of the remaining terms, the first factor has the value T. In this case, on the basis of (12), §2, it can be omitted in the product. *As a result, we obtain a formula which is equivalent to formula* (a) *and contains only variable elementary propositions.* This formula is the logical sum of distinct products of the form $X_1'X_2' \dots X_n'$, where X_i' denotes either X_i or $\bar{X}_i$. As can easily be seen from the above discussion, to every non-identically false function $\Phi(X_1, X_2, \dots, X_n)$, there corresponds a unique representation of this form.

If the function $\Phi(X_1, \dots, X_n)$ is not identically true, then its representation as a formula which does not contain constant elementary propositions can be obtained from formula (b) in an analogous manner. This second representation of the function $\Phi(X_1, \dots, X_n)$ will be dual to the first—as can very easily be observed.

EXAMPLES:

1. We consider the function $\Phi(X_1, X_2, X_3)$ of three variables; Φ takes on the value T if all the variables X_i assume the same value and F otherwise. Then $\Phi(T, T, T)$ and $\Phi(F, F, F)$ are T; and for other values of the variables, $\Phi(X_1, X_2, X_3)$ has the value F. We can represent this function in the following way:

$$X_1X_2X_3 \vee \bar{X}_1\bar{X}_2\bar{X}_3.$$

2. Suppose $\Phi(X_1, X_2, X_3, X_4)$ has the value T if, and only if, at least two of its arguments assume the value F. In this case it is convenient to utilize the second representation of the function. This representation will have the form:

$$(\bar{X}_1 \vee \bar{X}_2 \vee \bar{X}_3 \vee X_4) \mathbin{\&} (\bar{X}_1 \vee \bar{X}_2 \vee X_3 \vee \bar{X}_4) \mathbin{\&} (\bar{X}_1 \vee X_2 \vee \bar{X}_3 \vee \bar{X}_4) \mathbin{\&}$$
$$\mathbin{\&} (X_1 \vee \bar{X}_2 \vee \bar{X}_3 \vee \bar{X}_4) \mathbin{\&} (\bar{X}_1 \vee \bar{X}_2 \vee \bar{X}_3 \vee \bar{X}_4).$$

§6. Principal normal forms

In the preceding section we found, for an arbitrary non-identically false function $\Phi(X_1, X_2, \ldots, X_n)$ which depends on n variables, a formula from propositional algebra which represents this function and is the logical sum of distinct products of the form $X'_1X'_2 \ldots X'_n$, where X'_i denotes either X_i or $\bar{X}_i$. There, also, we remarked that *every non-identically false function* $\Phi(X_1, X_2, \ldots, X_n)$ *has a unique representation of the indicated form.* The disjunctive normal form is such a representation. We know, however, that one and the same function can be represented by various disjunctive (and also conjunctive) normal forms. We have thus found a method for choosing from various disjunctive normal forms, which represent a given function, a definite one which is the sum of terms of the form $X'_1 \ldots X'_n$. The disjunctive normal form obtained in this way will be called the *principal disjunctive normal form.* One can give another definition of the principal disjunctive normal form:

The principal disjunctive normal form of the formula $A(X_1, \ldots, X_n)$, *containing n and only n distinct variables, is the disjunctive normal form possessing the following properties:*

(a) *it does not contain two identical terms;*

(b) *none of the terms contains two identical factors;*

(c) *no term contains a variable simultaneously with its negation;*

(d) *every term contains either the variable* X_i *or its negation* $(i=1, 2, \ldots, n)$ *as a factor.*

It is easily seen that this definition is equivalent to the preceding. In fact, on the one hand, in virtue of (d), every term in the disjunctive normal form must contain n factors $X'_1, \ldots, X'_n$. On the other hand, in virtue of (b) and (c), this term cannot contain any other factor inasmuch as such a factor would be either X_i or $\bar{X}_i$ and then the term would already contain a factor which is

either X_i or $\bar{X}_i$. Thus our form consists of terms of the form $X_1', X_2', \ldots, X_n'$, all of which are distinct in virtue of (a). Therefore, the sum of these terms is that representation of the formula $A(X_1, \ldots, X_n)$ which we defined earlier.

Conditions (a), (b), (c), (d) are thus necessary and sufficient for the disjunctive normal form to be the principal normal form. Moreover, these conditions make it possible to express the rules which allow us to reduce any non-identically false formula to the principal disjunctive normal form. We shall describe these rules. Let $A(X_1, \ldots, X_n)$ be an arbitrary prescribed formula. We first reduce it to some disjunctive normal form. After that, if any term B does not contain the variable X_i, we replace it by the sum

$$X_i \,\&\, B \vee \bar{X}_i \,\&\, B.$$

This replacement is an equivalence transformation inasmuch as

$$X_i \,\&\, B \vee \bar{X}_i \,\&\, B \text{ is equivalent to } (X_i \vee \bar{X}_i) \,\&\, B.$$

But $X_i \vee \bar{X}_i$ is identically true and therefore

$$(X_i \vee \bar{X}_i) \,\&\, B \text{ is equivalent to } B;$$

consequently,

$$X_i \,\&\, B \vee \bar{X}_i \,\&\, B \text{ is equivalent to } B.$$

We can thus modify our normal form so that condition (d) is satisfied.

If identical terms appear in the expression obtained, then, deleting all except one of them, we again obtain an equivalent expression. If, after this, certain terms contain several identical factors, then the superfluous factors can be deleted. Finally, it is possible to delete all those terms which contain a variable simultaneously with its negation, inasmuch as such terms are identically false expressions. If all the terms were of this sort, this means that the entire sum is identically false. But then the formula A would also be false; in virtue of this, A does not have a principal disjunctive normal form. Therefore, if the formula A is not identically false, then in any of its disjunctive normal forms there must be terms which satisfy condition (c). After deleting all the terms containing some variable simultaneously with its negation, we obtain the disjunctive normal form of the formula A which satisfies conditions (a), (b), (c), (d), and which, consequently, is the principal normal form. We note that it is unnecessary for us to know in advance whether or not A is identically false. Performing the indicated operations, we decide this after having deleted all terms containing some variable together with its negation. If A is identically false, then all the terms will be deleted and we do not obtain the principal disjunctive normal form.

We define the principal conjunctive normal form analogously. This definition is carried out in terms which are dual to those we used in the definition of the principal disjunctive normal form.

The principal conjunctive normal form of the formula $A(X_1, \ldots, X_n)$ of n variables is the conjunctive normal form which is the product of distinct sums of the form $X_1' \vee X_2' \vee \ldots \vee X_n'$.

One can also state an equivalent definition in the form of conditions.

The principal conjunctive normal form of the formula $A(X_1, \ldots, X_n)$ *is its conjunctive normal form which satisfies the following conditions:*

(a′) *it does not contain two identical factors;*

(b′) *none of the factors contains two identical summands;*

(c′) *none of the factors contains a variable simultaneously with its negation;*

(d′) *every factor contains either* X_i *or* $\bar{X}_i$ *(for arbitrary* $i = 1, 2, \ldots, n$*) as a term.*

It is possible to prove that every non-identically true formula has a principal conjunctive normal form which is unique to within the order of arrangement of the factors and summands. The rules for the reduction of an arbitrary formula to the principal conjunctive normal form are analogous to those which we described for finding the principal disjunctive normal form, and they are expressed in the dual terms. The proof of all assertions concerning the principal conjunctive normal form can be obtained from the duality law or, of course, by carrying them out as we did for the principal disjunctive normal form.

Principal normal forms enable one to give a criterion for the equivalence of two arbitrary formulae A and B.

In fact, whatever the formulae A and B are, one can assume that they contain the same variables. If this were not the case and the formula A, say, did not contain the variable V, which occurs in B, then A could be replaced by the equivalent formula $A \,\&\, (V \vee \bar{V})$ which now contains the variable V. Thus, any two formulae can be replaced by formulae, equivalent to them, which contain the same variables. After this, these formulae must be reduced to the principal disjunctive or conjunctive normal forms. If A and B are equivalent formulae, then, in virtue of the uniqueness of the principal normal forms, the disjunctive as well as the conjunctive normal forms of these formulae must coincide completely. Thus, the comparison of the principal normal forms of the formulae A and B solves the problem concerning their equivalence.

Examples:

1. $X \vee Y(X \vee \bar{Y})$.

Removing the brackets, we obtain the disjunctive normal form of this formula:

$$X \vee YX \vee Y\bar{Y}.$$

However, the formula obtained is not the principal disjunctive normal form since the first summand does not contain the variable Y and the last summand contains the variable Y simultaneously with its negation. Replacing X by $XY \vee X\bar{Y}$, we obtain the formula

$$XY \vee X\bar{Y} \vee YX \vee Y\bar{Y}.$$

Deleting the last two summands, we obtain the formula

$$XY \vee X\bar{Y}.$$

This formula satisfies the conditions (a), (b), (c), (d), and, consequently, it is the principal disjunctive normal form of the formula $X \vee Y(X \vee \bar{Y})$.

The principal form just obtained allows us to simplify the given formula. Carrying X outside the brackets in the principal form, we obtain the formula $X(Y \vee \bar{Y})$, which is equivalent to the formula X.

2. Reduce the formula

$$(X \vee Y)(\bar{Y} \vee Z) \vee (X \vee \bar{Y})(Y \vee Z)$$

to the principal conjunctive normal form.

We reduce this formula to the conjunctive normal form by utilizing the second distributive law. We obtain the formula

$$(X \vee Y \vee X \vee \bar{Y})(X \vee Y \vee Y \vee Z)(\bar{Y} \vee Z \vee X \vee \bar{Y})(\bar{Y} \vee Z \vee Y \vee Z).$$

We delete the identically true factors and we omit any repetition of a summand in the remaining factors. We obtain

$$(X \vee Y \vee Z)(\bar{Y} \vee Z \vee X).$$

The formula thus obtained satisfies conditions (a′), (b′), (c′), (d′) and therefore it is the principal conjunctive normal form of the given formula. Using the second distributive law, we can simplify this formula further. Factoring out the summand $X \vee Z$, we obtain

$$X \vee Z \vee Y\bar{Y}.$$

Discarding the false summand $Y\bar{Y}$, we finally obtain

$$X \vee Z.$$

In concluding this chapter, we introduce further equivalence relations which are useful for the simplification of formulae:

$$X \vee XY \text{ is equivalent to } X. \tag{17}$$

$$X(X \vee Y) \text{ is equivalent to } X. \tag{18}$$

$$X \vee \bar{X}Y \text{ is equivalent to } X \vee Y. \tag{19}$$

$$\bar{X} \vee XY \text{ is equivalent to } \bar{X} \vee Y. \tag{20}$$

$$X(\bar{X} \vee Y) \text{ is equivalent to } XY. \tag{21}$$

$$\bar{X}(X \vee Y) \text{ is equivalent to } \bar{X}Y. \tag{22}$$

These relations can be formulated in the following way:

FIRST. *If a summand in a sum occurs as a factor in another summand, then the second summand can be deleted from the sum.*

SECOND. *If a factor in a product occurs as a summand in another factor, then the second factor can be deleted.*

THIRD and FOURTH. *In every summand, one can delete a factor which is equivalent to the negation of another summand.*

FIFTH and SIXTH. *In every factor, one can delete a summand if it is equivalent to the negation of another factor.*

EXAMPLES:

1. $X \vee XY \vee YZ \vee \bar{X}Z$.

We delete XY—on the basis of (17). In virtue of (19), we delete the factor $\bar{X}$ in the last summand. We obtain

$$X \vee YZ \vee Z.$$

By (17), we delete YZ and obtain

$$X \vee Z.$$

2. $(X \vee Y)(\bar{X}\bar{Y} \vee Z) \vee \bar{Z} \vee (X \vee Y)(U \vee V)$.

Since $\bar{X}\bar{Y}$ is equivalent to $\overline{X \vee Y}$, the summand $\bar{X}\bar{Y}$ can be deleted—on the basis of (21). We obtain

$$(X \vee Y)\, Z \vee \bar{Z} \vee (X \vee Y)(U \vee V).$$

Now, the factor Z can be deleted on the basis of (20). We obtain

$$(X \vee Y) \vee \bar{Z} \vee (X \vee Y)(U \vee V).$$

The last summand contains the factor $X \vee Y$ which coincides with the first summand. Therefore, on the basis of (17), the last summand can be deleted. We finally obtain

$$X \vee Y \vee \bar{Z}.$$

CHAPTER II

PROPOSITIONAL CALCULUS

§1. Concept of formula

The description of propositional algebra, which we gave in the preceding chapter, completely satisfies the requirements of rigor put forth in the Introduction as do all discussions concerning this system which we made there, inasmuch as they do not use at all the concept of actual infinity, i.e. they are constructive.

In fact, propositional algebra treats of finite configurations of symbols and the inter-relationships between them. These configurations of symbols are formulae containing letters and symbols of logical operations. The symbols T and F can be substituted for the letters, and the value which the formula then has can be calculated.

The number of letters in a formula is finite; the number of all possible distributions of the symbols T and F in this formula is also finite. The definitions of the logical operations also contain a finite number of conditions. On the other hand, we made statements about formulae only in terms of the indicated symbols and the inter-relationships among them, and therefore such propositions were free from any use of actual infinity.

Despite this fact, it is not always possible to apply propositional algebra directly to the statements of mathematics, without introducing, in this connection, the concept of actual infinity. In order for it to be always possible to do this, we must presuppose that every statement of mathematics is either true or false because this is precisely the sense of a proposition in propositional algebra. However, such a presupposition already depends on the concept of actual infinity. It is the law of excluded middle applied in this connection to infinite sets. In this form, this principle cannot be accepted in that part of mathematics which takes upon itself the task of establishing this law by showing that the utilization of it does not lead to a contradiction.

In this present chapter we consider an axiomatic logical system which in a definite sense is equivalent to propositional algebra. We shall call this system the propositional calculus.

It is necessary for us to give a sufficiently complete discussion of the propositional calculus in view of the fact that this calculus occurs as an integral part of all other logical calculi which we shall consider in the sequel.

The description of every calculus, as was asserted in the Introduction, includes a description of the symbols of this calculus, formulae—which are finite configurations of symbols—and after this the definition of true formulae.

The symbols of the propositional calculus consist of symbols of the following three categories:

1. Symbols of the first category are upper-case Latin letters A, B, $\ldots$, X, Y, Z and the same letters with indices A_1, A_2, $\ldots$. These symbols will be called propositional variables.

2. The second category consists of symbols which have the general name of logical connectives. There are four of these symbols:

$$\&, \vee, \rightarrow \text{ and } \text{—}.$$

They have the following names: the first is the conjunction symbol, the second is the disjunction symbol, the third is the implication symbol, and the last is the negation symbol.

3. The third category consists of a pair of symbols ( ), called brackets.

There are no other symbols in the propositional calculus than the ones just indicated.

Formulae of the propositional calculus are finite sequences of symbols of the categories just described. These sequences can be written in the form of a row of symbols.

To denote formulae, we shall usually use upper case Clarendon letters $\mathbf{A}$, $\mathbf{B}$, $\ldots$. These letters are not symbols of the calculus. They are only provisionally abbreviated notations for formulae.

Not every row of symbols is a formula. A complete definition of the concept of a formula is of a recursive character: one indicates certain initial formulae and also rules enabling one to form new formulae from these formulae. The meaning of this definition is that we are to understand as a formula those, and only those, rows which can be formed from the initial formulae by means of the consecutive application of the rules, indicated in the definition, for the formation of new formulae.

Definition of formulae:

(a) A propositional variable is a formula.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are formulae, then the rows

$$(\mathbf{A} \,\&\, \mathbf{B});$$
$$(\mathbf{A} \vee \mathbf{B});$$
$$(\mathbf{A} \rightarrow \mathbf{B});$$

and

$$\text{—}\,\mathbf{A}$$

are also formulae.

In this definition there appear in (a) the initial formulae which are here variable propositions. We shall call elementary propositions *elementary formulae.*

In (b), there appear rules enabling one to form new formulae from those obtained. Thus, the concept of a formula is completely defined.

We now introduce examples of formulae of the propositional calculus. The propositional variables A and B are formulae in virtue of (a). Consequently, on the basis of (b), the rows

$$(A \rightarrow B),$$
$$-A$$

are also formulae. Then, on the same basis,

$$(-A \;\&\; (A \rightarrow B)) \vee (A \;\&\; B)$$

is also a formula.

It is easy to see that the rows

$$(A \rightarrow -(A \;\&\; B)) \rightarrow ((C \vee D) \rightarrow (A \;\&\; B)),$$
$$(((A \;\&\; B) \;\&\; C) \;\&\; D),$$
$$(A \rightarrow (B \rightarrow (C \rightarrow D)))$$

are formulae.

On the other hand, the following rows of symbols are not formulae:

$$(A \;\&\; B; \qquad A - B; \qquad \&\, A;$$
$$B \rightarrow C; \qquad A \;\&\; B; \qquad A \vee B.$$

The first of these contains a single bracket. In the construction of formulae, single brackets are never introduced, in virtue of (b). Consequently, the set $(A \;\&\; B$ cannot arise in any act of forming a formula. It is easily seen that the second and third sets cannot be constructed on the basis of (b) in any way whatsoever. The fourth set is not a formula because, although A and B are formulae, the combination of formulae by means of the connective $\rightarrow$ is always accompanied by enclosure in brackets; the same can also be said of the last two sets.

It is clear from the definition of a formula that the construction of an arbitrary formula has the following characteristic.

We take some supply of elementary formulae or equivalently of propositional variables:

$$A_1^{(0)}, A_2^{(0)}, \ldots, A_{k_0}^{(0)}.$$

From them, we construct several formulae of the form

$$(A_i^{(0)} \rightarrow A_j^{(0)}); \quad (A_i^{(0)} \;\&\; A_j^{(0)}); \quad (A_i^{(0)} \vee A_j^{(0)}) \text{ and } -A_i^{(0)}.$$

We denote the formulae obtained by $A_1^{(1)}, \ldots, A_{k_1}^{(1)}$.

From the formulae $A_i^{(0)}$ and $A_j^{(1)}$ we construct, in the same way, new formulae $A_1^{(2)}, \ldots, A_{k_2}^{(2)}$, and so on. As a result, we obtain the formula $A_1^{(n)}$ which is the formula A.

All the formulae $A_j^{(i)}$ constructed in the indicated process are called *components of the formula A*.

The scheme, just introduced, for the construction of formulae is such that in the construction we complete the construction of all components of the formula also. Of course, a formula can also be given in another way. Since every formula is a finite sequence of symbols of the propositional calculus, it can be constructed by simply indicating which symbol is first in this sequence, which is second, and so on. Let us consider, for example, the row

$$((A \mathbin{\&} B) \rightarrow (C \vee D)).$$

It is completely defined by indicating in which order the symbols are taken. The first is (, the second A, the third &, and so on. However, not every row of symbols is a formula. We must still prove that a given row is a formula. In order to prove this, we must show that the row we have written down satisfies the definition of a formula. But, in the sense of the definition of formula, to prove that a given row is a formula means to construct it, starting from elementary formulae by means of the constructions indicated in (b).

In order to prove that the row we have introduced as an example is a formula, we must reason as follows. Since A and B are formulae, $(A \mathbin{\&} B)$ is also a formula. Since C and D are formulae, $(C \vee D)$ is also a formula. Since $(A \mathbin{\&} B)$ and $(C \vee D)$ are formulae, $((A \mathbin{\&} B) \rightarrow (C \vee D))$ is also a formula.

It is clear from this example that in order to prove that a given row is a formula, we have to carry out its construction according to the scheme indicated above so that in the construction of a formula we must also construct all its components.

We note, however, that the definition we gave of a component of a formula is not precise. A precise definition of a component of a formula is also of a recursive nature, connected with the operations utilized in the construction of formulae. To wit, we first state the definition of a component of elementary formulae and then, assuming that the components of the formulae $\boldsymbol{A}$ and $\boldsymbol{B}$ have already been determined, we determine the components of the formulae $(\boldsymbol{A} \mathbin{\&} \boldsymbol{B})$, $(\boldsymbol{A} \vee \boldsymbol{B})$, $(\boldsymbol{A} \rightarrow \boldsymbol{B})$ and $-\boldsymbol{A}$. We now introduce this definition of the concept of "component of a formula".

1. *A component of any elementary formula* (i.e. of a propositional variable) *is the formula itself.*

2. Let us assume the components of the formulae $\boldsymbol{A}$ and $\boldsymbol{B}$ have been determined. Components of the formula $(\boldsymbol{A} \mathbin{\&} \boldsymbol{B})$ will be all components of the formulae $\boldsymbol{A}$ and $\boldsymbol{B}$ together with the formula $(\boldsymbol{A} \mathbin{\&} \boldsymbol{B})$ itself. Components of the formulae $(\boldsymbol{A} \vee \boldsymbol{B})$, $(\boldsymbol{A} \rightarrow \boldsymbol{B})$ and $-\boldsymbol{A}$ are defined similarly.

It is easy to pick out the components of a formula directly from the formula. To this end we have brackets which indicate the sequence in which one must perform the operations in order to construct the formula.

We now introduce some changes in the mode of writing the formula.

We note that a formula can contain brackets enclosing all remaining symbols of the formula. For example,

$$(A \mathbin{\&} B).$$

Such brackets will be called exterior brackets. The row $A \mathbin{\&} B$, without brackets, will not appear in virtue of the definition of a formula. However, in order to abbreviate our writing, we will omit exterior brackets. In this connection, we do not modify the definition of a formula; we simply do not write certain symbols (in this case it is the exterior brackets), the presence of which in a sequence of symbols forming a formula is, of course, understood. We introduce another modification in the way of writing a formula with respect to the negation symbol. The formula $— A$ will be written in the form

$$\bar{A},$$

where, if A has exterior brackets, they will be omitted. After these modifications, the appearance of a formula in the propositional calculus will be the same as a formula in propositional algebra. Further, we make use of the same rules for omitting brackets as in propositional algebra. We shall assume that the connective & binds more strongly than all the remaining connectives and the connective $\vee$ more strongly than $\rightarrow$. In virtue of this rule, the formula

$$(A \mathbin{\&} B) \vee C$$

will be written in the form

$$A \mathbin{\&} B \vee C.$$

The formula

$$(A \vee B) \rightarrow (C \mathbin{\&} D)$$

will be written in the form

$$A \vee B \rightarrow C \mathbin{\&} D,$$

and so on.

EXAMPLES:

1. The formula $—((A \vee B) \rightarrow —(C \rightarrow D))$ is written in the modified transcription as

$$\overline{A \vee B \rightarrow \overline{C \rightarrow D}}.$$

2. The formula $(— A \vee (B \rightarrow (C \mathbin{\&} D)))$ is written in the form

$$\bar{A} \vee (B \rightarrow C \mathbin{\&} D).$$

We note that it would have been possible to give a definition of formula such that the configuration of its symbols would at once be of the same form as that which is obtained after the abbreviations are introduced. But then the definition itself of a formula would be more cumbersome and a formula would not be a simple sequence of symbols but rather a more complicated creation. On the other hand, for writing a formula, the simplifications we introduced are quite convenient.

§2. Definition of true formulae

The next step in the description of the propositional calculus will be the introduction of a certain class of formulae which will be called *true or deducible in the propositional calculus.* The definition of true formulae has the same recursive character as the definition of formula. We first define the initial true formulae and then we define the rules which enable us to construct new ones from the available true formulae. These rules will be called "*deduction rules*" and the initial true formulae will be called *axioms.* The formation of a true formula from the initial true formulae or axioms by means of applications of the deduction rules will be called a *deduction* of the given formula from axioms.

Axioms of the Propositional Calculus:

I

1. $A \to (B \to A)$.
2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

II

1. $A \& B \to A$.
2. $A \& B \to B$.
3. $(A \to B) \to ((A \to C) \to (A \to B \& C))$.

III

1. $A \to A \vee B$.
2. $B \to A \vee B$.
3. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$.

IV

1. $(A \to B) \to (\bar{B} \to \bar{A})$.
2. $A \to \bar{\bar{A}}$.
3. $\bar{\bar{A}} \to A$.

As is obvious from the above list, the axioms fall into four groups. Axioms of group I contain only the logical connective, implication. This connective also appears in all the remaining groups, but in group II logical multiplication is adjoined to implication; in group III, logical summation, and in group IV, negation, are adjoined.

Deduction rules

1. *Rule of substitution.* Let $\mathfrak{A}$ be a formula containing the letter A. If $\mathfrak{A}$ is a true formula of the propositional calculus, then, after replacing the letter A everywhere it occurs in it by an arbitrary formula $\mathfrak{B}$, we obtain a true formula.

2. *Rule of inference.* If A and $A \rightarrow B$ are true formulae in the propositional calculus, then B is also a true formula (this rule is traditionally called *modus parens*). By indicating the axioms and deduction rules, we have completely defined the concept of a true formula or of a formula which is deducible in the propositional calculus. Utilizing the deduction rules, we can, starting from the axioms, construct new true formulae and obtain, in this way, every true formula. We now consider some examples.

1. We shall show that the formula

$$(A \rightarrow B) \rightarrow (A \rightarrow A)$$

is true in the propositional calculus.

The formula

$$(A \rightarrow (B \rightarrow A)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow A))$$

is the result of substituting the variable A for C in axiom I.2. Since the antecedent of the implication obtained is axiom I.1, by applying the rule of inference, we find that

$$(A \rightarrow B) \rightarrow (A \rightarrow A)$$

is a true formula.

2. We shall show that

$$A \,\&\, B \rightarrow B \,\&\, A$$

is a true formula in the propositional calculus.

Let us perform a substitution in axiom II.3. Replacing A by the formula $A \,\&\, B$, we obtain the true formula

$$(A \,\&\, B \rightarrow B) \rightarrow ((A \,\&\, B \rightarrow C) \rightarrow (A \,\&\, B \rightarrow B \,\&\, C)).$$

We perform a substitution in this formula by replacing C by A. We obtain the true formula

$$(A \,\&\, B \rightarrow B) \rightarrow ((A \,\&\, B \rightarrow A) \rightarrow (A \,\&\, B \rightarrow B \,\&\, A)).$$

We see that the antecedent of this implication is axiom II.2. Therefore, applying the rule of inference, we obtain the true formula

$$(A \,\&\, B \rightarrow A) \rightarrow (A \,\&\, B \rightarrow B \,\&\, A).$$

The antecedent in this formula is axiom II.1. Applying the rule of inference, we obtain the required formula:

$$A \,\&\, B \rightarrow B \,\&\, A.$$

3. We shall show that

$$\bar{\bar{\bar{A}}} \rightarrow \bar{A}$$

is a true formula.

We make a substitution in axiom IV.1 by replacing B by formula $\bar{\bar{A}}$. We obtain the true formula

$$(A \rightarrow \bar{\bar{A}}) \rightarrow (\bar{\bar{\bar{A}}} \rightarrow \bar{A}).$$

The antecedent in this formula is axiom IV.2.

Applying the rule of inference, we obtain the required formula:

$$\bar{\bar{\bar{A}}} \rightarrow \bar{A}.$$

We make one remark regarding the rule of substitution. Although this rule is described perfectly clearly, its description cannot be considered completely satisfactory for if, in a formula, we replace some letter by a formula, we of course obtain some row of symbols but it still remains to prove that this row is a formula.

We now introduce another definition of the substitution rule which is equivalent to the old definition. It is much more cumbersome, but it will follow from it directly that the row obtained as a result of the substitution is still a formula. Moreover, this definition will be useful in other discussions.

The operation of substitution is a correspondence in which to every formula $\mathbf{A}$, for a given letter A and a prescribed formula $\mathbf{B}$, there corresponds a well-defined formula which we shall denote by $S_A^{\mathbf{B}}(\mathbf{A})$. The operation $S_A^{\mathbf{B}}(\mathbf{A})$ will be defined in the following way.

If $\mathbf{A}$ is a propositional variable A, then $S_A^{\mathbf{B}}(\mathbf{A})$ is the formula $\mathbf{B}$. If $\mathbf{A}$ is a propositional variable which is distinct from A, for example B, then $S_A^{\mathbf{B}}(\mathbf{A})$ is B.

We have, by the same token, defined the operation $S_A^{\mathbf{B}}(\mathbf{A})$ for all elementary formulae.

We shall assume that the formulae $S_A^{\mathbf{B}}(\mathbf{A}_1)$ and $S_A^{\mathbf{B}}(\mathbf{A}_2)$ are defined for the formulae $\mathbf{A}_1$ and $\mathbf{A}_2$.

We define the substitution operation for the formulae $\mathbf{A}_1 \,\&\, \mathbf{A}_2$, $\mathbf{A}_1 \vee \mathbf{A}_2$, $\mathbf{A}_1 \rightarrow \mathbf{A}_2$ and $\bar{\mathbf{A}}_1$ in the following way:

$$\begin{array}{ll}
S_A^{\mathbf{B}}(\mathbf{A}_1 \,\&\, \mathbf{A}_2) & \text{is the formula } S_A^{\mathbf{B}}(\mathbf{A}_1) \,\&\, S_A^{\mathbf{B}}(\mathbf{A}_2), \\
S_A^{\mathbf{B}}(\mathbf{A}_1 \vee \mathbf{A}_2) & \text{is the formula } S_A^{\mathbf{B}}(\mathbf{A}_1) \vee S_A^{\mathbf{B}}(\mathbf{A}_2), \\
S_A^{\mathbf{B}}(\mathbf{A}_1 \rightarrow \mathbf{A}_2) & \text{is the formula } S_A^{\mathbf{B}}(\mathbf{A}_1) \rightarrow S_A^{\mathbf{B}}(\mathbf{A}_2), \\
S_A^{\mathbf{B}}(\bar{\mathbf{A}}_1) & \text{is the formula } \overline{S_A^{\mathbf{B}}(\mathbf{A}_1)}.
\end{array}$$

We thus have a recursive definition of the operation $S_A^{\mathbf{B}}$ for all formulae of the propositional calculus. In this connection, the substitution operation $S_A^{\mathbf{B}}$ is also defined for formulae $\mathbf{A}$ which do not contain the variable A. In this case, as is easily seen, $S_A^{\mathbf{B}}(\mathbf{A})$ is $\mathbf{A}$. The substitution rule is now formulated in the following way.

If $\mathbf{A}$ is a true formula, then $S_A^{\mathbf{B}}(\mathbf{A})$ is also a true formula whatever the variable A and formula $\mathbf{B}$ are.

It follows directly from the new formulation of the substitution rule that the result of substitution in the formula $\mathbf{A}$ is also a formula.

We shall illustrate the operation $S_A^{\mathbf{B}}$ by an example.

We shall find

$$S_A^{A \rightarrow A \vee C}(A \,\&\, B \rightarrow (B \vee C) \,\&\, \overline{\bar{A} \rightarrow C}). \qquad (1)$$

According to the definition, this is the formula

$$S_A^{A \to A \vee C}(A \mathbin{\&} B) \to S_A^{A \to A \vee C}((B \vee C) \mathbin{\&} \overline{A \to C}).$$

$$S_A^{A \to A \vee C}(A \mathbin{\&} B) \text{ is } S_A^{A \to A \vee C}(A) \mathbin{\&} S_A^{A \to A \vee C}(B);$$

but

$$S_A^{A \to A \vee C}(A) \text{ is } A \to A \vee C,$$

and

$$S_A^{A \to A \vee C}(B) \text{ is } B.$$

Thus,

$$S_A^{A \to A \vee C}(A \mathbin{\&} B) \text{ is } (A \to A \vee C) \mathbin{\&} B.$$

Furthermore,

$$S_A^{A \to A \vee C}((B \vee C) \mathbin{\&} \overline{A \to C})$$

is

$$S_A^{A \to A \vee C}(B \vee C) \mathbin{\&} S_A^{A \to A \vee C}(\overline{A \to C});$$

$S_A^{A \to A \vee C}(B \vee C)$ is $B \vee C$ since this formula does not contain the variable A.

$$S_A^{A \to A \vee C}(\overline{A \to C}) \text{ is } \overline{S_A^{A \to A \vee C}(A \to C)},$$

and $S_A^{A \to A \vee C}(A \to C)$ is obviously $(A \to A \vee C) \to C$.

Thus,

$$S_A^{A \to A \vee C}(A \mathbin{\&} B \to (B \vee C) \mathbin{\&} \overline{A \to C})$$

is the formula

$$(A \to A \vee C) \mathbin{\&} B \to (B \vee C) \mathbin{\&} (\overline{A \to A \vee C) \to C}).$$

We see that as a result of the substitution under consideration the variable A in the formula

$$A \mathbin{\&} B \to (B \vee C) \mathbin{\&} \overline{A \to C}$$

was replaced by the formula

$$A \to A \vee C.$$

Besides the fundamental deduction rules, the substitution rule and the inference rule, we shall also have other rules for the formation of true formulae, which are derivatives of fundamental rules and are abbreviations of a repeated application of the fundamental rules. For all these rules we introduce a certain scheme which allows us to write them in abbreviated form.

These rules are usually expressed in the following terms.

"If the formulae $A, B, \ldots$ are true, then the formulae $M, N, \ldots$ are also true." We shall write this sort of definition in the form of the following scheme:

$$\frac{A, B, \ldots}{M, N, \ldots}.$$

Then the fundamental rules of deduction are written as follows:

Rule of substitution:

$$\frac{A}{S_A^B(A)}.$$

Rule of inference:

$$\frac{A, A \to B}{B}.$$

Performing consecutively a substitution in the different variables $A_1, \ldots, A_n$ of the formula A, where first A_1 is replaced by the formula B_1, then, in the resultant formula, A_2 is replaced by the formula B_2, and so on, we obtain, as a result, the formula

$$S_{A_n}^{B_n}(S_{A_{n-1}}^{B_{n-1}} \ldots (S_{A_1}^{B_1}(A))).$$

If the formulae B_i do not contain the variables $A_1, \ldots, A_n$, then the order in which we shall make these substitutions is immaterial, but in the contrary case this is not so. We introduce the following operation. We replace the variables $A_1, \ldots, A_n$ in the formula $A(A_1, \ldots, A_n)$ by variables which do not occur in any of the formulae $B_1, \ldots, B_n$. Suppose these are the variables $X_1, \ldots, X_n$. We then obtain the formula $A(X_1, \ldots, X_n)$. After this, we perform the successive substitutions:

$$S_{X_n}^{B_n}(\ldots(S_{X_1}^{B_1}(A(X_1, \ldots, X_n)))).$$

We obtain a formula concerning which one can say that it is the result of the simultaneous replacement of the variables $A_1, \ldots, A_n$ by the formulae $B_1, \ldots, B_n$, respectively. The operation obtained will be denoted by

$$S_{A_1 \ldots A_n}^{B_1 \ldots B_n}(A).$$

The formula $A(X_1, \ldots, X_n)$ plays an auxiliary role in this operation. If we do not replace the variables A_i by the variables X_i, then, in the successive substitutions, there will also be a replacement of the variables A_i occurring in the formulae B_j; then the result of the operation will not be a simultaneous replacement of the variables A_i in the formula A. The formula $A(X_1, \ldots, X_n)$ is called the *indicative formula*. The operation

$$S_{A_1 \ldots A_n}^{B_1 \ldots B_n}(A)$$

will be called *compound substitution in the formula A* or simply *substitution in the formula A*. If $n=1$, the indicative formula is not needed inasmuch as in this case we are dealing with the usual application of the substitution rule. Therefore, if the formula A is valid in the propositional calculus, then the formula

$$S_{A_1 \ldots A_n}^{B_1 \ldots B_n}(A)$$

is also a valid formula, and we obtain the rule:

$$\frac{A}{S_{A_1 \ldots A_n}^{B_1 \ldots B_n}(A)}.$$

The second derived rule is applied to formulae of the form

$$A_1 \to (A_2 \to (\ldots (A_{n-1} \to A_n) \ldots))$$

and is expressed as follows: *if the formulae*

$$A_1, \ldots, A_{n-1} \textit{ and } A_1 \to (A_2 \to (\ldots(A_{n-1} \to A_n)\ldots))$$

are true, then the formula A_n is also true in the propositional calculus. This assertion is easily proved by successive application of the inference rule.

In fact, if

$$A_1 \text{ and } A_1 \to (A_2 \to (\ldots(A_{n-1} \to A_n)\ldots))$$

are true formulae, then

$$A_2 \to (\ldots(A_{n-1} \to A_n)\ldots)$$

is also a true formula.

Since

$$A_2 \text{ and } A_2 \to (A_3 \to (\ldots(A_{n-1} \to A_n)\ldots)$$

are true formulae,

$$A_3 \to (\ldots(A_{n-1} \to A_n)\ldots)$$

is also a true formula.

Continuing this line of reasoning further, we finally find that A_n is a true formula in the propositional calculus.

We thus obtain a new rule which we shall call the *compound inference rule.* It is written down in the form of the scheme

$$\frac{A_1, A_2, \ldots, A_{n-1}, A_1 \to (A_2 \to (\ldots(A_{n-1} \to A_n)\ldots))}{A_n}.$$

We shall say that a formula is false if its negation is true. We shall denote any formulae which are true in the propositional calculus by the letter T and any false formulae by the letter F.

THEOREM 1. *$B \to T$ is a true formula.*

We perform a substitution in axiom I.1; replacing A by T, we obtain

$$T \to (B \to T).$$

Since T is a true formula, applying the inference rule to the formulae

$$T \text{ and } T \to (B \to T),$$

we obtain

$$B \to T.$$

THEOREM 2. *$A \to A$ is a true formula.*

We replace C by A in axiom I.2; we obtain

$$(A \to (B \to A)) \to ((A \to B) \to (A \to A)).$$

The antecedent of this formula is axiom I.1. Applying the rule of inference, we have

$$(A \to B) \to (A \to A).$$

Replacing B by T, we obtain

$$(A \to T) \to (A \to A).$$

On the basis of the preceding theorem, $A \to T$ is a true formula. Applying the rule of inference, we find that

$$A \to A$$

is a true formula.

3. The deduction theorem

Instead of deducing the formulae which are true in the propositional calculus from the axioms, we shall exhibit a much shorter way, applying the rule of inference directly, by proving the so-called deduction theorem. This theorem allows us to establish the deducibility of various formulae in a much simpler way than the direct deduction of these formulae.

We shall say that *the formula B is deducible from the formulae $A_1, \ldots, A_n$ if the formula B can be deduced by using only the law of inference, taking $A_1, \ldots, A_n$ and all formulae which are true in the propositional calculus as the initial formulae.*

A precise definition of the deducibility of the formula B from the formulae $A_1, \ldots, A_n$, which are called the initial formulae, is formulated as follows:

(a) *Every formula $A_i (1 \leqslant i \leqslant n)$ is deducible from the formulae*

$$A_1, \ldots, A_n.$$

(b) *Every formula which is true in the propositional calculus is deducible from*

$$A_1, \ldots, A_n.$$

(c) *If the formulae A and $A \to B$ are deducible from $A_1, \ldots, A_n$, then the formula B is also deducible from*

$$A_1, \ldots, A_n.$$

The assertion that the formula B is deducible from $A_1, \ldots, A_n$ will be denoted by

$$A_1, \ldots, A_n \vdash B.$$

The deducibility of formula B from formulae $A_1, \ldots, A_n$ differs from the deducibility of a true formula from the axioms of the propositional calculus in this that in the second case the substitution rule occurs among the rules of inference but it does not in the first case. We could also have defined a deduction of the formula B from the formulae $A_1, \ldots, A_n$ in which the substitution rule holds. If we used the concept of deducibility in this and the other sense, it would then have been necessary to differentiate these two concepts by distinct terms. However, in view of the fact that we shall be dealing only with the deducibility of B from other formulae in one sense, viz. that corresponding to the given definition, we shall not introduce a new term for this concept. If we encounter the deducibility of the formula B from the formulae $A_1, \ldots, A_n$ in the other sense, we shall state this explicitly. We note that if $A_1, \ldots, A_n$ are axioms or other true formulae, then the class

of formulae deducible from them coincides with the class of all true formulae inasmuch as every true formula is assumed to be deducible from an arbitrary system of formulae.

We shall extend the meaning of the expression

$$A_1, \ldots, A_n \vdash B \tag{1}$$

to the case when there are no formulae A_i at all (i.e. when $n=0$), by assuming that then B is simply a true formula in the propositional calculus. In this case, expression (1) naturally transforms into

$$\vdash B.$$

All further discussion on the deducibility of formula B from the formulae $A_1, \ldots, A_n$ will be extended to this case also.

THE DEDUCTION THEOREM. *If the formula B is deducible from the formulae $A_1, \ldots, A_n$, then*

$$A_1 \to (A_2 \to (\ldots(A_n \to B)\ldots))$$

is a true formula.

We shall first prove that if

$$A_1, A_2, \ldots, A_n \vdash B,$$

then

$$A_1, A_2, \ldots, A_{n-1} \vdash A_n \to B.$$

We shall prove this assertion by induction in the following way. We shall prove that it is true if B is any one of the formulae A_i or a true formula in the propositional calculus. Next, we shall show that if our assertion is true for the formulae B' and $B' \to B''$, then it is also true for B''.

If B coincides with the formula A_i, then either $i = n$ or $i < n$. In the first case, $A_n \to A_n$ is a true formula; it is obtained by a substitution in the formula $A \to A$ (see Theorem 2, §2). Therefore,

$$A_1, \ldots, A_{n-1} \vdash A_n \to A_n.$$

Let us assume that $i < n$; then by substitution in axiom I.1, we obtain

$$\vdash A_i \to (A_n \to A_i).$$

The last formula, being true, is deducible from the formulae $A_1, \ldots, A_{n-1}$. But formula A_i is also deducible from the formulae $A_1, \ldots, A_{n-1}$. Therefore, the formula $A_n \to A_i$ is deducible from the formulae $A_1, \ldots, A_{n-1}$.

The case when B is a true formula is obvious.

We shall now assume that $A_n \to B'$ and $A_n \to (B' \to B'')$ are deducible from $A_1, \ldots, A_{n-1}$. The formula

$$(A_n \to (B' \to B'')) \to ((A_n \to B') \to (A_n \to B''))$$

is true in the propositional calculus inasmuch as it is obtained by a substitution in axiom I.2. Therefore, this formula is deducible from $A_1, \ldots, A_{n-1}$.

Both antecedents of this formula are deducible from $A_1, \ldots, A_{n-1}$, by assumption. Applying the rule of inference twice, we obtain the formula $A_n \to B''$ which consequently is also deducible from the formulae $A_1, \ldots, A_{n-1}$.

We have thus proved that if

$$A_1, \ldots, A_n \vdash B,$$

then

$$A_1, \ldots, A_{n-1} \vdash A_n \to B. *$$

* The proof is by induction on the number of applications of the inference rule in the derivation of B from $A_1, A_2 \ldots, A_n$. Assume that the deduction theorem holds with not more than some number k of applications of the rule of inference, then we show that it holds for $k+1$ applications. For suppose that some formula B'' is derived from $A_1, A_2, \ldots, A_n$ by $k+1$ applications of the rule of inference; let the last of these be the derivation of B'' from the formulae B' and $B' \to B''$. Necessarily B' and $B' \to B''$ are derived from $A_1, A_2, \ldots, A_n$ by not more than k applications of the rule of inference, and so by assumption the deduction theorem holds for B' and $B' \to B''$, and hence by the foregoing proof it holds also for B'', which proves the deduction theorem for derivations which use $k+1$ applications of the rule of inference, But the deduction theorem certainly holds for zero applications of the inference rule, for then B must be one of $A_1, A_2, \ldots, A_n$ and so by induction the deduction theorem holds for any number of applications of the rule of inference.

[R. L. G.]

Our line of reasoning, in accordance with what was stated above, is also applicable to the case when $n = 1$. In this case, our assertion consists in the following: if $A_n \vdash B$, then $\vdash A_n \to B$.

After what has been proved, we can easily establish the validity of the deduction theorem. We shall assume that

$$A_1, \ldots, A_n \vdash B$$

holds. In this case, we shall have that

$$A_1, \ldots, A_{n-1} \vdash A_n \to B.$$

Applying the same thesis a second time, we obtain

$$A_1, \ldots, A_{n-2} \vdash A_{n-1} \to (A_n \to B).$$

Reasoning this way further, we finally obtain that

$$A_1 \vdash A_2 \to (A_3 \to (\ldots \to (A_n \to B)\ldots)).$$

But the same line of reasoning can be applied once more giving

$$\vdash (A_1 \to (A_2 \to (A_3 \to (\ldots \to (A_n \to B)\ldots)))),$$

which concludes the proof of the deduction theorem.

§4. Derived rules of the propositional calculus

THEOREM 1. $\vdash (A \to B) \to ((B \to C) \to (A \to C))$.

We consider the formulae $A \to B$, $B \to C$ and A. One can deduce the formula C from these formulae with the aid of the rule of inference only.

In this case, we conclude, on the basis of the deduction theorem, that

$$(A \to B) \to ((B \to C) \to (A \to C))$$

is a true formula. This completes the proof of the theorem.

We make a substitution in this formula—replacing A, B, C by the formulae $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, respectively obtaining the true formula

$$(\boldsymbol{A} \to \boldsymbol{B}) \to ((\boldsymbol{B} \to \boldsymbol{C}) \to (\boldsymbol{A} \to \boldsymbol{C})).$$

If the formulae $\boldsymbol{A} \to \boldsymbol{B}$ and $\boldsymbol{B} \to \boldsymbol{C}$ turn out to be true, then, applying the compound rule of inference to the last formula, we find that the formula $\boldsymbol{A} \to \boldsymbol{C}$ is also true. We thus obtain a rule which is written as follows:

$$\frac{\boldsymbol{A} \to \boldsymbol{B},\ \boldsymbol{B} \to \boldsymbol{C}}{\boldsymbol{A} \to \boldsymbol{C}}.$$

This rule is called the *syllogism rule*.

THEOREM 2. $\vdash (A \to (B \to C)) \to (B \to (A \to C))$.

We consider the formulae $A \to (B \to C)$, B, A. Applying the rule of inference twice, we find that

$$A \to (B \to C), B, A \vdash C.$$

Applying the deduction theorem, we obtain the required formula. From the truth of this formula, we deduce, the same way as in the preceding theorem, the rule

$$\frac{\boldsymbol{A} \to (\boldsymbol{B} \to \boldsymbol{C})}{\boldsymbol{B} \to (\boldsymbol{A} \to \boldsymbol{C})}.$$

This rule is called the *rule for interchanging antecedents.*

THEOREM 3. $\vdash A \to (B \to A \,\&\, B)$.

We shall first prove that

$$(\boldsymbol{T} \to A) \to ((\boldsymbol{T} \to B) \to (\boldsymbol{T} \to A \,\&\, B)), A, B \vdash A \,\&\, B$$

(as we have agreed, $\boldsymbol{T}$ denotes an arbitrary true formula). For the sake of brevity, we shall denote the first of these formulae by A^0. The formula $A \to (\boldsymbol{T} \to A)$ is true. Therefore, the formula $\boldsymbol{T} \to A$ is deducible from the formula A and, *a fortiori*, from the formulae A^0, A, B. We thus have

$$A^0, A, B \vdash \boldsymbol{T} \to A.$$

Using an analogous line of reasoning, we see that

$$A^0, A, B \vdash \boldsymbol{T} \to B.$$

From A^0, by applying the rule of inference three times, we obtain the formula $A \,\&\, B$, from which it follows that

$$A^0, A, B \vdash A \,\&\, B.$$

Applying the deduction theorem, we have

$$\vdash A^0 \to (A \to (B \to A \& B)).$$

But A^0 is a true formula; it can be obtained by a substitution in axiom II.3 (p. 48). From this we obtain

$$\vdash A \to (B \to A \& B),$$

and the theorem is proved. This theorem implies the following rule:

$$\frac{A, B}{A \& B}.$$

The converse rule,

$$\frac{A \& B}{A, B},$$

follows from axioms II.1 and II.2. These two rules imply the rule

$$\frac{A \& B}{B \& A}$$

which can, moreover, also be deduced from the formula

$$\vdash A \& B \to B \& A,$$

whose validity was proved (see §2, p. 49).

THEOREM 4. $\vdash F \to A$.

Replacing B by T in axiom IV.1, we obtain that

$$(A \to T) \to (\bar{T} \to \bar{A});$$

but $A \to T$ is a true formula (see Theorem 1, §2).

Applying the rule of inference, we obtain the formula

$$\vdash \bar{\bar{F}} \to \bar{A},$$

where, using the definition of a false formula, we have replaced T by $\bar{F}$. Substituting $\bar{A}$ for A, we obtain that

$$\vdash \bar{\bar{F}} \to \bar{\bar{A}}.$$

By means of the syllogism rule, from the last formula and axioms IV.2 and IV.3, we obtain

$$\vdash F \to A.$$

THEOREM 5.

$$\vdash (A \to (B \to C)) \to (A \& B \to C), \qquad \text{(a)}$$

$$\vdash (A \& B \to C) \to (A \to (B \to C)). \qquad \text{(b)}$$

We shall prove the first assertion. We have

$$A \to (B \to C), A \& B \vdash C.$$

In fact, the formulae

$$A \& B \to A \quad \text{and} \quad A \& B \to B$$

are axioms and therefore they are true in the propositional calculus. Therefore formulae A and B are deducible from the formulae $A \to (B \to C)$ and $A \,\&\, B$.

Applying the rule of inference to the formulae

$$A \quad \text{and} \quad A \to (B \to C)$$

and then once more to the formulae

$$B \quad \text{and} \quad B \to C,$$

we conclude that C is deducible from the formulae $A \to (B \to C)$, $A \,\&\, B$. From this, on the basis of the deduction theorem, follows the validity of assertion (a).

We now prove assertion (b).

We consider the system of formulae

$$A \,\&\, B \to C, A, B.$$

We shall show that that formula C is deducible from this system. On the basis of Theorem 3, we have that

$$\vdash A \to (B \to A \,\&\, B).$$

From this it follows that the formula $A \,\&\, B$ is deducible from the formulae

$$A \,\&\, B \to C, A, B.$$

In this case, C is also deducible from the same formulae. Applying the deduction theorem, we shall prove assertion (b) of the theorem.

The following two rules follow from the theorem just proved:

$$\frac{A \to (B \to C)}{A \,\&\, B \to C} \quad \text{and} \quad \frac{A \,\&\, B \to C}{A \to (B \to C)}.$$

The first of these rules will be called the *rule for the combination of antecedents.*

THEOREM 6. $\vdash \mathbf{A} \,\&\, \bar{\mathbf{A}} \to \mathbf{F}$.

From axiom I.1, by means of a substitution, we obtain

$$\vdash A \to (T \to A).$$

Moreover, from axiom IV.1, we obtain

$$\vdash (T \to A) \to (\bar{A} \to F)$$

where we have kept in mind that $\bar{T}$ is F.

We apply the syllogism rule to these two formulae and obtain:

$$\vdash A \to (\bar{A} \to F),$$

from which by the rule for the combination of antecedents it follows that

$$\vdash A \,\&\, \bar{A} \to F.$$

Finally, a substitution yields the formula sought:

$$\vdash \mathbf{A} \,\&\, \bar{\mathbf{A}} \to \mathbf{F}.$$

§5. Monotonicity

We shall say that the formula $\boldsymbol{A}$ is stronger than the formula $\boldsymbol{B}$ if $\vdash \boldsymbol{A} \to \boldsymbol{B}$ (though it would perhaps be better to say "not weaker than").

DEFINITION. The formula $\boldsymbol{A}(A)$, containing the variable A, is said to be monotonically increasing (or monotonically decreasing) with respect to the variable A if $\boldsymbol{B}_1 \to \boldsymbol{B}_2$ implies $\boldsymbol{A}(\boldsymbol{B}_1) \to \boldsymbol{A}(\boldsymbol{B}_2)$ (or, if $\boldsymbol{B}_1 \to \boldsymbol{B}_2$ implies $\boldsymbol{A}(\boldsymbol{B}_2) \to \boldsymbol{A}(\boldsymbol{B}_1)$, respectively). Here, as always in the sequel, $\boldsymbol{A}(\boldsymbol{B}_1)$ and $\boldsymbol{A}(\boldsymbol{B}_2)$ denote formulae which are obtained from $\boldsymbol{A}(A)$ by replacing a propositional variable A by the formulae $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, respectively.

The concept of monotonicity owes its significance to the following theorem.

THEOREM 1. *All fundamental logical operations are monotonic with respect to all variables occurring in them: the formulae $A \,\&\, B$ and $A \vee B$ increase monotonically with respect to A and B, $\bar{A}$ decreases with respect to A, $A \to B$ decreases with respect to A and increases with respect to B.*

Proof. 1. The formula $A \,\&\, B$ increases monotonically with respect to A and B. In fact, let

$$\vdash \boldsymbol{B}_1 \to \boldsymbol{B}_2. \tag{a}$$

By a substitution in axiom II.1, we obtain

$$\vdash \boldsymbol{B}_1 \,\&\, B \to \boldsymbol{B}_1. \tag{b}$$

It follows from these formulae, by the syllogism rule, that

$$\vdash \boldsymbol{B}_1 \,\&\, B \to \boldsymbol{B}_2. \tag{c}$$

By substitutions in axiom II.3 ($\boldsymbol{B}_1 \,\&\, B$ for A, $\boldsymbol{B}_2$ for B and B for C), we obtain

$$\vdash (\boldsymbol{B}_1 \,\&\, B \to \boldsymbol{B}_2) \to [(\boldsymbol{B}_1 \,\&\, B \to B) \to (\boldsymbol{B}_1 \,\&\, B \to \boldsymbol{B}_2 \,\&\, B)]. \tag{d}$$

The formulae $\boldsymbol{B}_1 \,\&\, B \to B$ and $\boldsymbol{B}_1 \,\&\, B \to \boldsymbol{B}_2$ are true. Therefore, applying the rule of inference twice to (d), we obtain

$$\vdash \boldsymbol{B}_1 \,\&\, B \to \boldsymbol{B}_2 \,\&\, B.$$

This proves that the formula $A \,\&\, B$ increases monotonically with respect to A. That it increases monotonically with respect to B is proved in an analogous manner.

2. The formula $A \vee B$ increases monotonically with respect to A and B. We shall prove, for example, that $A \vee B$ increases monotonically with respect to B.

Let

$$\vdash \boldsymbol{B}_1 \to \boldsymbol{B}_2. \tag{a}$$

By a substitution, we obtain from axiom III.2 that

$$\vdash \boldsymbol{B}_2 \to A \vee \boldsymbol{B}_2. \tag{b'}$$

Now the syllogism rule yields

$$\vdash \boldsymbol{B}_1 \to A \vee \boldsymbol{B}_2. \tag{c'}$$

Moreover, substitution in III.1 yields

$$\vdash A \vdash A \vee \boldsymbol{B}_2. \tag{c''}$$

We now perform a substitution in axiom III.3, putting $\boldsymbol{B}_1$ for B and $A \vee \boldsymbol{B}_2$ for C. We obtain

$$\vdash (A \to A \vee \boldsymbol{B}_2) \to [(\boldsymbol{B}_1 \to A \vee \boldsymbol{B}_2) \to (A \vee \boldsymbol{B}_1 \to A \vee \boldsymbol{B}_2)]. \tag{d'}$$

We apply the rule of inference twice to (d'), making use of (c') and (c''). We obtain

$$\vdash A \vee \boldsymbol{B}_1 \to A \vee \boldsymbol{B}_2.$$

3. The formula $\bar{A}$ decreases monotonically with respect to A. This follows directly from axiom IV.1.

4. The formula $A \to B$ increases monotonically with respect to B and decreases monotonically with respect to A.

We shall prove that the formula $A \to B$ decreases monotonically with respect to A. Let

$$\vdash \boldsymbol{B}_1 \to \boldsymbol{B}_2. \tag{a}$$

In virtue of the monotonicity of the formula $A \,\&\, B$, proved above, we have

$$\vdash (\boldsymbol{B}_2 \to B) \,\&\, \boldsymbol{B}_1 \to (\boldsymbol{B}_2 \to B) \,\&\, \boldsymbol{B}_2. \tag{e}$$

Further, from the truth of the formula $A \to A$ (see p. 53), we obtain, by means of a substitution, that

$$\vdash (\boldsymbol{B}_2 \to B) \to (\boldsymbol{B}_2 \to B)$$

and, applying the rule for the combination of antecedents,

$$\vdash (\boldsymbol{B}_2 \to B) \,\&\, \boldsymbol{B}_2 \to B. \tag{f}$$

With the aid of the syllogism rule, we obtain from (e) and (f),

$$\vdash (\boldsymbol{B}_2 \to B) \,\&\, \boldsymbol{B}_1 \to B.$$

If we now apply the rule converse to the rule for the combination of antecedents to the formula just obtained, we obtain what is required:

$$\vdash (\boldsymbol{B}_2 \to B) \to (\boldsymbol{B}_1 \to B).$$

We shall now prove that the formula $A \to B$ increases monotonically with respect to B.

Let $\vdash \boldsymbol{B}_1 \to \boldsymbol{B}_2$. Analogous to formula (f), we obtain

$$\vdash (A \to \boldsymbol{B}_1) \,\&\, A \to \boldsymbol{B}_1. \tag{f'}$$

From (a) and (f'), by the syllogism rule, we have

$$\vdash (A \to \boldsymbol{B}_1) \,\&\, A \to \boldsymbol{B}_2,$$

from which, applying the rule converse to the rule for the combination of antecedents, we obtain

$$\vdash (A \to B_1) \to (A \to B_2),$$

which was required to be proved.

It follows from the very definition of monotonicity that if the formula $A(B)$ increases monotonically with respect to its part B and if

$$\vdash A(B_1),$$

then, replacing B_1 by the weaker formula B_2, we obtain a true formula also. But if $A(B)$ decreases with respect to B, then $A(B_1)$ remains true upon replacing B_1 by a stronger formula.

These considerations frequently allow us to simplify proofs in which one performs a replacement of a part of any formula by a weaker or stronger formula. In such cases it is usually sufficient to verify whether or not one has the required monotonicity.

§6. Equivalent formulae

A formula having the form

$$(A \to B) \,\&\, (B \to A)$$

will be written briefly in the form

$$A \sim B.$$

The symbol $\sim$ is not a symbol of the propositional calculus, but is used only to abbreviate the expression indicated above. Besides, using it offers many advantages. (We could have introduced the symbol $\sim$ directly into the propositional calculus in a completely adequate manner for the system we are considering, but then we would eventually have had to introduce further new axioms and this would have complicated the calculus without giving any advantages.)

We shall call the symbol $\sim$ the symbol of equivalence and the formula $A \sim B$ will be called an equivalence.

As an example, we shall consider the expression

$$A \sim \overline{\overline{A}}.$$

Axioms IV.2 and IV.3 have the form

$$A \to \overline{\overline{A}} \quad \text{and} \quad \overline{\overline{A}} \to A.$$

Applying the rule

$$\frac{A, B}{A \,\&\, B}$$

to it, we obtain

$$\vdash (A \to \overline{\overline{A}}) \,\&\, (\overline{\overline{A}} \to A)$$

or

$$\vdash A \sim \overline{\overline{A}}$$

if we write this expression in the new form.

We now introduce the important concept of *equivalent formulae.* We shall say that the formulae A and B are equivalent if

$$\vdash A \sim B$$

holds.

We note at once that any two true formulae in the propositional calculus are equivalent.

The relation of equivalence of formulae is symmetric, i.e. if A is equivalent to B, then B is also equivalent to A. In fact, if

$$\vdash A \sim B$$

holds, or, expressed otherwise,

$$\vdash (A \rightarrow B) \,\&\, (B \rightarrow A),$$

then, in virtue of the rule

$$\frac{A \,\&\, B}{A, B},$$

we have that

$$\vdash A \rightarrow B \quad \text{and} \quad \vdash B \rightarrow A.$$

Applying the converse rule

$$\frac{A, B}{A \,\&\, B},$$

we obtain

$$\vdash (B \rightarrow A) \,\&\, (A \rightarrow B),$$

or

$$\vdash B \sim A.$$

The relation of equivalence of formulae is also transitive, i.e. if A is equivalent to B and B is equivalent to C, then A is also equivalent to C.

In fact, it follows from $\vdash A \sim B$ and $\vdash B \sim C$, on the basis of

$$\frac{A \,\&\, B}{A, B},$$

that

$$\vdash A \rightarrow B; \ \vdash B \rightarrow C.$$

Applying the syllogism rule to these formulae, we obtain

$$\vdash A \rightarrow C.$$

In virtue of the symmetry of equivalence, we have

$$\vdash C \rightarrow A.$$

And, finally, applying the rule

$$\frac{A, B}{A \,\&\, B},$$

we obtain

$$\vdash A \sim C.$$

In order to prove the validity of the equivalence

$$A \sim B,$$

it is sufficient to prove the following two implications:

$$\boldsymbol{A} \to \boldsymbol{B} \quad \text{and} \quad \boldsymbol{B} \to \boldsymbol{A};$$

and, conversely, these two implications are true if the equivalence $\boldsymbol{A} \sim \boldsymbol{B}$ is true.

The truth of this fact is easily seen from what we stated above. The same situation can be written in the form of two rules:

$$\frac{\boldsymbol{A} \to \boldsymbol{B}, \boldsymbol{B} \to \boldsymbol{A}}{\boldsymbol{A} \sim \boldsymbol{B}}, \quad \frac{\boldsymbol{A} \sim \boldsymbol{B}}{\boldsymbol{A} \to \boldsymbol{B}, \boldsymbol{B} \to \boldsymbol{A}}.$$

If use is made of a concept we introduced earlier, we can formulate this assertion as follows:

A necessary and sufficient condition that the formulae $\boldsymbol{A}$ and $\boldsymbol{B}$ be equivalent is that $\boldsymbol{A}$ be stronger than $\boldsymbol{B}$ and $\boldsymbol{B}$ be stronger than $\boldsymbol{A}$.

EXAMPLE. It follows directly from the equivalence

$$A \sim \overline{\overline{A}}$$

proved above that the formulae A and $\overline{\overline{A}}$ are equivalent.

EQUIVALENCE THEOREM. *Suppose the formula $\boldsymbol{A}(A)$ contains the proposition variable A and that the formulae $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are equivalent. Then the formulae $\boldsymbol{A}(\boldsymbol{B}_1)$ and $\boldsymbol{A}(\boldsymbol{B}_2)$, obtained from $\boldsymbol{A}(A)$ upon replacing A by $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ respectively, are also equivalent.*

More precisely:

$$\vdash (\boldsymbol{B}_1 \sim \boldsymbol{B}_2) \to [\boldsymbol{A}(\boldsymbol{B}_1) \sim \boldsymbol{A}(\boldsymbol{B}_2)]. \tag{I}$$

The proof will be carried out by induction, on the number of operations, with the aid of which the formula $\boldsymbol{A}(A)$ is constructed.

If the number of these operations is zero, then the formula $\boldsymbol{A}(A)$ is simply A and therefore formula (I) has the form

$$(\boldsymbol{B}_1 \sim \boldsymbol{B}_2) \to (\boldsymbol{B}_1 \sim \boldsymbol{B}_2).$$

This formula is true inasmuch as it is obtained by a substitution in the formula $A \to A$ deduced earlier.

Now, let the assertion (I) be satisfied for all formulae which are obtained by no more than n operations and let $\boldsymbol{A}(A)$ be constructed by $n + 1$ operations. The last operation participating in the construction of $\boldsymbol{A}(A)$ is one of the four possible operations: &, V, $\to$ or —. Therefore formula $\boldsymbol{A}(A)$ has one of the following forms:

$$\boldsymbol{A}_1(A) \,\&\, \boldsymbol{A}_2(A),\ \boldsymbol{A}_1(A) \vee \boldsymbol{A}_2(A),\ \boldsymbol{A}_1(A) \to \boldsymbol{A}_2(A),\ \overline{\boldsymbol{A}_1(A)},$$

where the formulae $\boldsymbol{A}_1(A)$ and $\boldsymbol{A}_2(A)$ are already constructed by no more than n operations.

It follows from this that the theorem will be proved if the assertions

1. $\vdash (B_1 \sim B_2) \to [A_1(B_1) \& A_2(B_1) \sim A_1(B_2) \& A_2(B_2)]$,
2. $\vdash (B_1 \sim B_2) \to [A_1(B_1) \vee A_2(B_1) \sim A_1(B_2) \vee A_2(B_2)]$,
3. $\vdash (B_1 \sim B_2) \to \{[A_1(B_1) \to A_2(B_1)] \sim [A_1(B_2) \to A_2(B_2)]\}$,
4. $\vdash (B_1 \sim B_2) \to [\overline{A_1(B_1)} \sim \overline{A_1(B_2)}]$

are proved under the assumption that

$$\vdash (B_1 \sim B_2) \to [A_1(B_1) \sim A_1(B_2)] \tag{a}$$

and

$$\vdash (B_1 \sim B_2) \to [A_2(B_1) \sim A_2(B_2)] \tag{b}$$

are satisfied [in case 4, it is necessary to hypothesize (a) only].

We shall thus prove 1-4.

1. By means of substitutions in axiom II.1, we obtain

$$\vdash A_1(B_1) \& A_2(B_1) \to A_1(B_1). \tag{1}$$

On the other hand, from (a), which is given, and which can be written in the form

$$\vdash (B_1 \sim B_2) \to \{[A_1(B_1) \to A_1(B_2)] \& [A_1(B_2) \to A_1(B_1)]\}$$

by the application of axiom II.1 and the syllogism rule, we obtain

$$\vdash (B_1 \sim B_2) \to [A_1(B_1) \to A_1(B_2)],$$

from which, using the rule for the interchange of antecedents, we obtain

$$\vdash A_1(B_1) \to [(B_1 \sim B_2) \to A_1(B_2)]. \tag{2}$$

By the syllogism rule, we obtain from (1) and (2)

$$\vdash A_1(B_1) \& A_2(B_1) \to [(B_1 \sim B_2) \to A_1(B_2)],$$

from which it follows, if the antecedents are interchanged, that

$$\vdash (B_1 \sim B_2) \to [A_1(B_1) \& A_2(B_1) \to A_1(B_2)]. \tag{3}$$

In an entirely analogous manner one can obtain

$$\vdash (B_1 \sim B_2) \to [A_1(B_1) \& A_2(B_1) \to A_2(B_2)]. \tag{4}$$

Performing substitutions in axiom II.3 and applying the deduction rule twice [on the basis of (3) and (4)], we obtain

$$\vdash (B_1 \sim B_2) \to [A_1(B_1) \& A_2(B_1) \to \\ \to A_1(B_2)] \& [A_1(B_1) \& A_2(B_1) \to A_2(B_2)]. \tag{5}$$

On the other hand, applying the rule for the addition of antecedents to axiom II.3 and performing the appropriate substitutions, we obtain

$$\vdash [A_1(B_1) \& A_2(B_1) \to A_1(B_2)] \& [A_1(B_1) \& A_2(B_1) \to \\ \to A_2(B_2)] \to [A_1(B_1) \& A_2(B_1) \to A_1(B_2) \& A_2(B_2)]. \tag{6}$$

Application of the syllogism rule to (5) and (6) yields

$$\vdash (B_1 \sim B_2) \to [A_1(B_1) \,\&\, A_2(B_1) \to A_1(B_2) \,\&\, A_2(B_2)]. \tag{7}$$

From (7) and from

$$\vdash (B_1 \sim B_2) \to [A_1(B_2) \,\&\, A_2(B_2) \to A_1(B_1) \,\&\, A_2(B_1)], \tag{8}$$

which is proved analogously, we obtain, upon applying axiom II.3,

$$\vdash (B_1 \sim B_2) \to [A_1(B_1) \,\&\, A_2(B_1) \sim A_1(B_2) \,\&\, A_2(B_2)].$$

2. It follows from the definition of equivalence and axiom II.1 that

$$\vdash (A \sim B) \to (A \to B). \tag{9}$$

Moreover,

$$\vdash B \to B \vee C \tag{10}$$

holds (axiom III.1).

Further, from Theorem 1, §4 by means of a substitution, we obtain

$$\vdash (A \to B) \to [(B \to B \vee C) \to (A \to B \vee C)]$$

from which by interchanging antecedents and applying the deduction rule [on the basis of (10)] we obtain

$$\vdash (A \to B) \to (A \to B \vee C). \tag{11}$$

Applying the syllogism rule to (9) and (11), we have

$$\vdash (A \sim B) \to (A \to B \vee C).$$

Substitutions in the last expression yield

$$\vdash [A_1(B_1) \sim A_1(B_2)] \to [A_1(B_1) \to A_1(B_2) \vee A_2(B_2)],$$

from which, using (a), by the syllogism rule we obtain

$$\vdash (B_1 \sim B_2) \to [A_1(B_1) \to A_1(B_2) \vee A_2(B_2)]. \tag{12}$$

It is proved analogously that

$$\vdash (B_1 \sim B_2) \to [A_2(B_1) \to A_1(B_2) \vee A_2(B_2)]. \tag{13}$$

Applying axiom II.3, we obtain from (12) and (13) that

$$\vdash (B_1 \sim B_2) \to [A_1(B_1) \to A_1(B_2) \vee A_2(B_2)] \,\&\, \\ \&\, [A_2(B_1) \to A_1(B_2) \vee A_2(B_2)]. \tag{14}$$

On the other hand, applying the rule for the addition of antecedents to axiom III.3 and performing substitutions, we obtain

$$\vdash [A_1(B_1) \to A_1(B_2) \vee A_2(B_2)] \,\&\, [A_2(B_1) \to A_1(B_2) \vee A_2(B_2)] \to \\ \to [A_1(B_1) \vee A_2(B_1) \to A_1(B_2) \vee A_2(B_2)]. \tag{15}$$

Applying the syllogism rule to (14) and (15), we obtain

$$\vdash (B_1 \sim B_2) \to [A_1(B_1) \vee A_2(B_1) \to A_1(B_2) \vee A_2(B_2)].$$

It follows from this and the assertion

$$\vdash (B_1 \sim B_2) \to [A_1(B_2) \vee A_2(B_2) \to A_1(B_1) \vee A_2(B_1)],$$

which is proved analogously, according to axiom II.3, that

$$\vdash (B_1 \sim B_2) \rightarrow [A_1(B_1) \vee A_2(B_1) \sim A_1(B_2) \vee A_2(B_2)].$$

3. From (a), with the application of axiom II.2 and the syllogism rule, we obtain

$$\vdash (B_1 \sim B_2) \rightarrow [A_1(B_2) \rightarrow A_1(B_1)]. \tag{16}$$

In an analogous manner, from (b), applying axiom II.1 and the syllogism rule, we obtain

$$\vdash (B_1 \sim B_2) \rightarrow [A_2(B_1) \rightarrow A_2(B_2)]. \tag{17}$$

Interchanging antecedents in (16) and (17), we obtain

$$\vdash A_1(B_2) \rightarrow [(B_1 \sim B_2) \rightarrow A_1(B_1)] \tag{18}$$

and

$$\vdash A_2(B_1) \rightarrow [(B_1 \sim B_2) \rightarrow A_2(B_2)]. \tag{19}$$

Adding the antecedents in (18), we obtain

$$\vdash A_1(B_2) \;\&\; (B_1 \sim B_2) \rightarrow A_1(B_1). \tag{20}$$

On the other hand, substitution in the formula

$$\vdash A \rightarrow A$$

yields

$$\vdash [A_1(B_1) \rightarrow A_2(B_1)] \rightarrow [A_1(B_1) \rightarrow A_2(B_1)],$$

from which we obtain, by interchanging antecedents, that

$$\vdash A_1(B_1) \rightarrow \{[A_1(B_1) \rightarrow A_2(B_1)] \rightarrow A_2(B_1)\}. \tag{21}$$

By applying the syllogism rule to (20) and (21), we obtain

$$\vdash A_1(B_2) \;\&\; (B_1 \sim B_2) \rightarrow \{[A_1(B_1 \rightarrow A_2(B_1)] \rightarrow A_2(B_1)\},$$

from which by the rule for the addition of antecedents it follows that

$$\vdash [A_1(B_2) \;\&\; (B_1 \sim B_2)] \;\&\; [A_1(B_2) \rightarrow A_2(B_1)] \rightarrow A_2(B_1). \tag{22}$$

According to the syllogism rule, (22) and (19) imply

$$\vdash [A_1(B_2) \;\&\; (B_1 \sim B_2)] \;\&\; [A_1(B_1) \rightarrow A_2(B_1)] \rightarrow \\ \rightarrow [(B_1 \sim B_2) \rightarrow A_2(B_2)]. \tag{23}$$

Applying the rule converse to the rule for the addition of antecedents to (23), we obtain

$$\vdash [A_1(B_2) \;\&\; (B_1 \sim B_2)] \rightarrow \{[A_1(B_1) \rightarrow A_2(B_1)] \rightarrow [(B_1 \sim B_2) \rightarrow A_2(B_2)]\},$$

from which we obtain, by applying this rule once more and interchanging the antecedents, that

$$\vdash (B_1 \sim B_2) \rightarrow (A_1(B_2) \rightarrow \{[A_1(B_1) \rightarrow A_2(B_1)] \rightarrow [(B_1 \sim B_2) \rightarrow A_2(B_2)]\}).$$

The formula

$$\vdash (B_1 \sim B_2) \rightarrow ([A_1(B_1) \rightarrow A_2(B_1)] \rightarrow \{A_1(B_2) \rightarrow [(B_1 \sim B_2) \rightarrow A_2(B_2)]\}) \tag{24}$$

is obtained from the preceding one by an application of the rule for the

interchange of antecedents and the syllogism rule. In exactly the same way, from (24) we obtain

$$\vdash (B_1 \sim B_2) \to ((B_1 \sim B_2) \to \{[A_1(B_1) \to A_2(B_1)] \to [A_1(B_2) \to A_2(B_2)]\}), \quad (25)$$

by making use of the rule for the interchange of antecedents (twice), the syllogism rule (twice), and the monotonicity of implication. From (25), by adding antecedents and utilizing the formula

$$\vdash A \to A \mathbin{\&} A, \tag{26}$$

we obtain

$$\vdash (B_1 \sim B_2) \to \{[A_1(B_1) \to A_2(B_1)] \to [A_1(B_2) \to A_2(B_2)]\}. \tag{27}$$

[Formula (26) can be obtained by replacing all variables in axiom II.3 by the letter A.]

Exactly as (27) one proves that

$$\vdash (B_1 \sim B_2) \to \{[A_1(B_2) \to A_2(B_2)] \to [A_1(B_1) \to A_2(B_1)]\}. \tag{28}$$

It follows from (27) and (28) (by axiom II.3) that

$$\vdash (B_1 \sim B_2) \to \{[A_1(B_1) \to A_2(B_1)] \sim [A_1(B_2) \to A_2(B_2)]\}.$$

4. From (a) and from

$$\vdash [A_1(B_1) \to A_1(B_2)] \to [\overline{A_1(B_2)} \to \overline{A_1(B_1)}],$$

which is obtained by a substitution in axiom IV.1, we obtain assertion 4 by the syllogism rule:

$$\vdash (B_1 \sim B_2) \to [\overline{A_1(B_1)} \sim \overline{A_1(B_2)}].$$

This completes the proof of the equivalence theorem.

We note that neither in the formulation nor in the proof of this theorem did we assume or use the fact that the replacement of A by B_1 and B_2 respectively in the formula $A(A)$ occurs everywhere where A occurs in $A(A)$. In obtaining the formulae $A(B_1)$ and $A(B_2)$ it is necessary only that in every place where A occurs in A such a replacement by B_1 and B_2 either occurs or does not occur simultaneously.

This observation allows us to formulate the result just proved as follows:

If, in the formula A, we replace any part B_1 of it by an equivalent formula B_2, then the newly obtained formula $A(B_2)$ is equivalent to the former, i.e.

$$\vdash (B_1 \sim B_2) \to [A(B_1) \sim A(B_2)].$$

§7. Some theorems on deducibility

THEOREM 1. $\vdash (A \sim T) \sim A$.

Proof. According to the definition of equivalence we must prove the following:

$$\vdash [(A \sim T) \to A] \mathbin{\&} [A \to (A \sim T)].$$

But, on the basis of Theorem 3, §4, Chapter II, it is sufficient to this end to prove that

$$\vdash (A \sim T) \rightarrow A \tag{1}$$

and

$$\vdash A \rightarrow (A \sim T). \tag{2}$$

We shall first prove (1). By the definition of deducibility,

$$T \rightarrow A \vdash A,$$

and therefore

$$\vdash (T \rightarrow A) \rightarrow A \tag{3}$$

on the basis of the deduction theorem.

It follows from axiom II.2 that

$$\vdash [(A \rightarrow T) \,\&\, (T \rightarrow A)] \rightarrow (T \rightarrow A),$$

i.e.

$$\vdash (A \sim T) \rightarrow (T \rightarrow A). \tag{4}$$

By the syllogism rule, we conclude from (3) and (4) that

$$\vdash (A \sim T) \rightarrow A. \tag{1}$$

We shall now prove (2). We have $\vdash T$ and, *a fortiori*, that $A \vdash T$; the deduction theorem then yields $\vdash A \rightarrow T$ and, *a fortiori*, that $A \vdash A \rightarrow T$. We apply the deduction theorem a second time and obtain

$$\vdash A \rightarrow (A \rightarrow T). \tag{5}$$

From axiom I.1, we have that

$$\vdash A \rightarrow (T \rightarrow A). \tag{6}$$

By means of substitutions in axiom II.3, we obtain

$$\vdash [A \rightarrow (A \rightarrow T)] \rightarrow \{[A \rightarrow (T \rightarrow A)] \rightarrow [A \rightarrow (A \rightarrow T) \,\&\, (T \rightarrow A)]\}.$$

Utilizing (5) and (6) and applying the rule of inference twice, we obtain

$$\vdash A \rightarrow (A \rightarrow T) \,\&\, (T \rightarrow A),$$

i.e.

$$\vdash A \rightarrow (A \sim T), \tag{2}$$

which was to be proved.

THEOREM 2. $\vdash (A \sim F) \sim \bar{A}$.

Proof. We shall first prove that

$$\vdash (A \sim F) \rightarrow \bar{A}.$$

Substitution in axiom IV.1 yields

$$\vdash (A \rightarrow F) \rightarrow (\bar{F} \rightarrow \bar{A}),$$

i.e.

$$\vdash (A \rightarrow F) \rightarrow (T \rightarrow \bar{A}). \tag{7}$$

Moreover, it was proved in the course of the proof of Theorem 1, above, that

$$\vdash (T \rightarrow A) \rightarrow A.$$

Substituting $\bar{A}$ for A, we obtain

$$\vdash (T \to \bar{A}) \to \bar{A}. \tag{8}$$

From (7) and (8) it follows by the syllogism rule that

$$\vdash (A \to F) \to \bar{A}. \tag{9}$$

By means of substitutions in axiom II.1, we obtain

$$\vdash (A \to F) \,\&\, (F \to A) \to (A \to F). \tag{10}$$

From (9) and (10) by the syllogism rule we obtain

$$\vdash (A \to F) \,\&\, (F \to A) \to \bar{A},$$

i.e.

$$\vdash (A \sim F) \to \bar{A}. \tag{11}$$

We shall now prove that

$$\vdash \bar{A} \to (A \sim F),$$

i.e.

$$\vdash \bar{A} \to (A \to F) \,\&\, (F \to A). \tag{12}$$

By means of substitutions in axioms I.1 and IV.1, we obtain

$$\vdash \bar{A} \to (T \to \bar{A}) \tag{13}$$

and

$$\vdash (T \to \bar{A}) \to (\bar{\bar{A}} \to \bar{T}). \tag{14}$$

Since $\bar{\bar{A}} \sim A$, then, by the equivalence theorem, the formula obtained by replacing $\bar{\bar{A}}$ by A is also valid:

$$\vdash (T \to \bar{A}) \to (A \to \bar{T}),$$

i.e.

$$\vdash (T \to \bar{A}) \to (A \to F) \tag{15}$$

(in virtue of the fact that $\bar{T}$ is F). By the syllogism rule, we deduce from (13) and (15) that

$$\vdash \bar{A} \to (A \to F). \tag{16}$$

Since $\vdash F \to A$ (by Theorem 4, §4, Chapter II), we have $\bar{A} \vdash F \to A$, and, by the deduction theorem, that

$$\vdash \bar{A} \to (F \to A). \tag{17}$$

By means of substitutions in axiom II.3, we obtain

$$\vdash [\bar{A} \to (A \to F)] \to \{[\bar{A} \to (F \to A)] \to [\bar{A} \to (A \to F) \,\&\, (F \to A)]\}.$$

Taking (16) and (17) into consideration and applying the rule of inference twice, we obtain

$$\vdash \bar{A} \to (A \to F) \,\&\, (F \to A),$$

i.e.

$$\vdash \bar{A} \to (A \sim F). \tag{18}$$

Finally, substituting formula (11) in place of A and formula (18) in place of

B in the true formula $\vdash A \to (B \to A \,\&\, B)$ (see Theorem 3, §4, Chapter II) and using the rule of inference twice, we obtain

$$\vdash [(A \sim F) \to \bar{A}] \,\&\, [\bar{A} \to (A \sim F)],$$

i.e.

$$\vdash (A \sim F) \sim \bar{A},$$

which is what we were required to prove.

THEOREM 3. $\vdash \mathcal{A}(T) \,\&\, \mathcal{A}(F) \to [(A \sim T) \to \mathcal{A}(A)] \,\&\, [(A \sim F) \to \mathcal{A}(A)]$.

Proof. In virtue of the equivalence theorem, we have

$$\vdash (A \sim B) \to [\mathcal{A}(A) \sim \mathcal{A}(B)]. \tag{19}$$

From axiom II.2, we have

$$\vdash [\mathcal{A}(A) \sim \mathcal{A}(B)] \to [\mathcal{A}(B) \to \mathcal{A}(A)]. \tag{20}$$

According to the syllogism rule, (19) and (20) imply

$$\vdash (A \sim B) \to [\mathcal{A}(B) \to \mathcal{A}(A)]$$

or, interchanging the antecedents,

$$\vdash \mathcal{A}(B) \to [(A \sim B) \to \mathcal{A}(A)]. \tag{21}$$

Substituting T and F in place of B in (21), we obtain

$$\vdash \mathcal{A}(T) \to [(A \sim T) \to \mathcal{A}(A)] \tag{22}$$

and

$$\vdash \mathcal{A}(F) \to [(A \sim F) \to \mathcal{A}(A)], \tag{23}$$

respectively. But in virtue of axioms II.1 and II.2, we have

$$\vdash \mathcal{A}(T) \,\&\, \mathcal{A}(F) \to \mathcal{A}(T) \tag{24}$$

and

$$\vdash \mathcal{A}(T) \,\&\, \mathcal{A}(F) \to \mathcal{A}(F). \tag{25}$$

Applying the syllogism rule to the two pairs of formulae (22), (24) and (23), (25), we obtain respectively

$$\vdash \mathcal{A}(T) \,\&\, \mathcal{A}(F) \to [(A \sim T) \to \mathcal{A}(A)]$$

and

$$\vdash \mathcal{A}(T) \,\&\, \mathcal{A}(F) \to [(A \sim F) \to \mathcal{A}(A)],$$

from which, by axiom II.3, we have

$$\vdash \mathcal{A}(T) \,\&\, \mathcal{A}(F) \to [(A \sim T) \to \mathcal{A}(A)] \,\&\, [(A \sim F) \to \mathcal{A}(A)],$$

which is what we were required to prove.

THEOREOM 4.

$$\vdash [(A \sim T) \to \mathcal{A}(A)] \,\&\, [(A \sim F) \to \mathcal{A}(A)] \to [(A \sim T) \vee (A \sim F) \to \mathcal{A}(A)].$$

We obtain the proof directly by making a substitution in axiom III.3,

$$\vdash [(A \sim T) \to \mathcal{A}(A)] \to \{[(A \sim F) \to \mathcal{A}(A)] \to$$
$$\to [(A \sim T) \vee (A \sim F) \to \mathcal{A}(A)]\}$$

and applying the rule for the addition of antecedents to the formula thus obtained.

THEOREM 5. $\vdash A \vee \bar{A}$.

Proof. By a substitution, we obtain from axioms III.1 and III.2 that

$$\vdash A \to A \vee \bar{A},$$
$$\vdash \bar{A} \to A \vee \bar{A}.$$

Further, applying axiom IV.1 and the rule of inference, we obtain

$$\vdash \overline{A \vee \bar{A}} \to \bar{A},$$
$$\vdash \overline{A \vee \bar{A}} \to \bar{\bar{A}}.$$

By means of substitutions in axiom II.3 we obtain

$$\vdash (\overline{A \vee \bar{A}} \to \bar{\bar{A}}) \to ((\overline{A \vee \bar{A}} \to \bar{A}) \to (\overline{A \vee \bar{A}} \to \bar{\bar{A}} \,\&\, \bar{A})),$$

from which, by applying the rule of inference twice, we have

$$\vdash \overline{A \vee \bar{A}} \to \bar{\bar{A}} \,\&\, \bar{A}.$$

In view of the equivalence of A and $\bar{\bar{A}}$, we have

$$\vdash \overline{A \vee \bar{A}} \to A \,\&\, \bar{A}.$$

Applying the syllogism rule to the last formula and to the assertion of Theorem 6, §4, we find

$$\vdash \overline{A \vee \bar{A}} \to F.$$

Substitutions in axiom IV.1 yield

$$\vdash (\overline{A \vee \bar{A}} \to F) \to (F \to \overline{\overline{A \vee \bar{A}}}),$$

from which, applying the rule of inference, we obtain

$$\vdash F \to \overline{\overline{A \vee \bar{A}}}.$$

Since F is T and the formula $\overline{\overline{A \vee \bar{A}}}$ is equivalent to $A \vee \bar{A}$, we have

$$\vdash T \to A \vee \bar{A}.$$

Since T is deducible, by applying the rule of inference we find that

$$\vdash A \vee \bar{A},$$

which is what we were required to prove.

THEOREM 6. $\vdash (A \sim T) \vee (A \sim F)$.

The proof is obtained from Theorem 5 upon application of Theorems 1 and 2.

THEOREM 7. $\vdash A(T) \,\&\, A(F) \rightarrow A(A)$.

Proof. Applying the syllogism rule to the formulae proved in Theorems 3 and 4, we obtain

$$\vdash A(T) \,\&\, A(F) \rightarrow [(A \sim T) \vee (A \sim F) \rightarrow A(A)].$$

From this, using the rule for the interchange of antecedents and the rule of inference, we obtain

$$\vdash A(T) \,\&\, A(F) \rightarrow A(A),$$

which is what we were required to prove.

We now introduce an abbreviated notation which will be needed in the sequel.

Suppose the formula A contains precisely n propositions:

$$A = A(A_1, A_2, \ldots, A_n).$$

We define by induction the formula

$$\prod_{\delta_1, \ldots, \delta_n = T, F} A(\delta_1, \ldots, \delta_n). \tag{*}$$

When $n = 1$, then $\prod\limits_{\delta_1 = T, F} A(\delta_1)$ is the formula $A(T) \,\&\, A(F)$. Suppose formulae (*) are defined for all A for which $n \leqslant k$. Now let the formula

$$A(A_1, A_2, \ldots, A_k, A_{k+1})$$

contain precisely $k + 1$ propositions $A_1, \ldots, A_k, A_{k+1}$; we shall denote the formula

$$[\prod_{\delta_1, \ldots, \delta_k = T, F} A(\delta_1, \ldots, \delta_k, T)] \,\&\, [\prod_{\delta_1, \ldots, \delta_k = T, F} A(\delta_1, \ldots, \delta_k, F)]$$

by

$$\prod_{\delta_1, \ldots, \delta_{k+1} = T, F} A(\delta_1, \ldots, \delta_k, \delta_{k+1}).$$

We have thus defined formula (*) for $A(A_1, \ldots, A_k, A_{k+1})$, containing $k + 1$ propositions, by formula (*) for $A(A_1, \ldots, A_k, T)$ and $A(A_1, \ldots, A_k, F)$, containing k variable propositions each.

Formula (*) can essentially be defined as the logical product of all possible formulae obtained from $A(A_1, \ldots, A_n)$ by means of all possible replacements of the variable propositions $A_1, A_2, \ldots, A_n$ with the aid of T and F. We have, however, not yet proved that such a product is associative and commutative, i.e. does not depend on the manner in which the brackets are placed and the order of the factors. We have therefore had to define formula (*) by giving some definite order to the factors and a definite manner of placing the brackets.

Starting with the last theorem, it is easy to prove (by induction) the following assertion.

THEOREM 8. $\vdash \prod\limits_{\delta_1, \ldots, \delta_n = T, F} A(\delta_1, \ldots, \delta_n) \rightarrow A(A_1, A_2, \ldots, A_n)$.

§8. Formulae in propositional algebra and in the propositional calculus

Formulae in the propositional calculus can be interpreted as formulae in propositional algebra. To this end, we shall treat free variables in the propositional calculus as variables in propositional algebra, i.e. variables in the formal sense, taking on the values T and F. We define the operations &, V, $\rightarrow$ and — as in propositional algebra; then every formula, for arbitrary values of the variables, will itself assume one of the values T or F, calculated according to propositional algebra.

We now perform on an arbitrary formula A of the propositional calculus a replacement of the variable propositions occurring in it by the formulae $\boldsymbol{T}$ and $\boldsymbol{F}$. If in the same formula A, considered as a formula in propositional algebra, the variable propositions are assigned the values T and F, whereby we assign the value T (or F) to those variables which in A, considered as a formula in the propositional calculus, were replaced respectively by $\boldsymbol{T}$ (or $\boldsymbol{F}$), then the formula A in the propositional calculus assumes one of the values: T or F. We shall prove that if in this connection the value of A is T (respectively F), then under the corresponding replacement in A, as a formula in the propositional calculus, we obtain

$$\vdash A(\boldsymbol{T}, \boldsymbol{F}) \sim \boldsymbol{T} \text{ [respectively: } \vdash A(\boldsymbol{T}, \boldsymbol{F}) \sim \boldsymbol{F}].$$

It is sufficient to prove our assertion for the simplest formulae, obtained from proposition variables with the aid of one of the operations: &, V, $\rightarrow$ or —; in fact, then this assertion for an arbitrary formula is proved by induction on the basis of the equivalence theorem. Thus, we prove our assertion for the simplest formulae:

(a) $A \rightarrow B$; (c) $A \vee B$;

(b) $A \,\&\, B$; (d) $\bar{A}$.

For each of these formulae it is necessary to consider all possible replacements of the variables by the formulae $\boldsymbol{T}$ and $\boldsymbol{F}$; however, we shall consider only some cases inasmuch as the proof in the remaining cases is carried out analogously.

We first consider formula (d). It is necessary to prove:

$$(\mathrm{d}_1) \vdash \overline{\boldsymbol{F}} \sim \boldsymbol{T},$$

$$(\mathrm{d}_2) \vdash \overline{\boldsymbol{T}} \sim \boldsymbol{F}.$$

Assertion (d_1) follows from the definition of a false formula and from the fact that any two true formulae are equivalent in the propositional calculus.

(d_2) $\vdash \boldsymbol{F} \rightarrow \overline{\boldsymbol{T}}$ is obtained from Theorem 4, §4. We shall prove

$$\vdash \overline{\boldsymbol{T}} \rightarrow \boldsymbol{F}.$$

Substitutions in axiom IV.1 yield

$$\vdash (\boldsymbol{F} \rightarrow \boldsymbol{T}) \rightarrow (\overline{\boldsymbol{T}} \rightarrow \overline{\boldsymbol{F}}).$$

The antecedent of this formula is deducible in virtue of (d_1). Applying the rule of inference, we find that

$$\vdash \bar{T} \to \bar{\bar{F}}.$$

Since from axiom IV.13 we have

$$\vdash \bar{\bar{F}} \to F,$$

then, applying the syllogism rule, we obtain that

$$\vdash \bar{T} \to F.$$

(a) Here we shall consider all possible replacements and prove that

$$(a_1)\ \vdash (T \to F) \sim F,$$

$$(a_2)\ \vdash (F \to T) \sim T,$$

$$(a_3)\ \vdash (T \to T) \sim T,$$

$$(a_4)\ \vdash (F \to F) \sim T.$$

(a_1) It suffices to prove that

$$\vdash F \to (T \to F) \tag{1}$$

and that

$$\vdash (T \to F) \to F. \tag{2}$$

But we obtain (1) directly by a substitution in the formula $\vdash F \to A$, which was proved in §4.

In order to prove (2), we perform a substitution in axiom IV.1:

$$\vdash (T \to F) \to (\bar{F} \to \bar{T}),$$

from which by an interchange of antecedents we obtain

$$\vdash \bar{F} \to [T \to F) \to \bar{T}].$$

Noting now that $\vdash \bar{F} \sim T$ and that $\vdash \bar{T} \sim F$, and applying the equivalence theorem (see page 64) and the rule of inference, we obtain the required result:

$$\vdash (T \to F) \to F.$$

In the remaining three cases, the proof follows directly from the fact that the formulae $F \to T$, $T \to T$ and $F \to F$ are valid. In fact, we obtain $\vdash F \to T$ by means of a substitution in the valid formula

$$\vdash F \to A,$$

which was proved in §4; and the validity of the formulae $T \to T$ and $F \to F$ follows from Theorem 2, §2 which asserts that $\vdash A \to A$.

We consider further, as an example, formula (c): $A \vee B$ and one of the possible cases of replacement, viz: we shall prove that

$$\vdash T \vee F \sim T.$$

Here, $\vdash T \vee F \rightarrow T$ follows from Theorem 1, §2, whereas

$$\vdash T \rightarrow T \vee F$$

is the result of a substitution in axiom III.1.

The proofs in the remaining cases are carried out in a similar manner and can be left to the reader as exercises.

§9. Consistency of the propositional calculus

The problem of consistency arises in the consideration of any calculus; it is one of the cardinal problems in mathematical logic. We shall now give the definition of the consistency of a logical calculus, which definition pertains not only to the propositional calculus but also to all logical systems studied in mathematical logic.

We shall say that a logical calculus is consistent if no two formulae are deducible in it one of which is the negation of the other.

In other words, a consistent calculus is a calculus such that whatever the formula A may be, the formulae A and $\bar{A}$ can never be deduced simultaneously from the axioms of this calculus with the aid of the rules of this calculus.

The consistency problem consists in the following: Is the given calculus consistent or not.

If, in the calculus, deducible formulae A and $\bar{A}$ can be found in the calculus, then such a calculus is called *inconsistent*. Such calculi are of no value. All logical systems which are of any importance whatsoever are such that if any of them turns out to be inconsistent then this would signify that all formulae are deducible in it and therefore such systems are not capable of reflecting the difference between truth and falsity.

For example, if in the propositional calculus certain formulae A and $\bar{A}$ turned out to be deducible, then, in virtue of the formulae we have proved above (see Theorems 4 and 6, §4)

$$\vdash F \rightarrow A$$

and

$$\vdash A \,\&\, \bar{A} \rightarrow F,$$

we would have that

$$\vdash A \,\&\, \bar{A} \rightarrow A.$$

But if A and $\bar{A}$ are deducible, then, in virtue of the rule

$$\frac{A, B}{A \,\&\, B},$$

we have that $A \,\&\, \bar{A}$ is also deducible, and, consequently, the formula A is also deducible in the propositional calculus. But if the propositional variable A is deducible, then by means of a substitution into it we can deduce an arbitrary formula.

Everything stated above about the propositional calculus turns out to be true also for all those logical calculi which we shall consider in the sequel.

The situation that every formula is deducible in an inconsistent calculus can be used in the proof of consistency. To this end it suffices to show that there exists at least one non-deducible formula. The consistency of the calculus will follow from this. The consistency of the propositional calculus can be established, however, very simply even without this.

THEOREM. *The propositional calculus is consistent.*

Proof. As we have already stated above, every formula in the propositional calculus can be considered at the same time as a formula in propositional algebra.

We shall show that all formulae which are deducible in the propositional calculus and considered as formulae in the propositional algebra are identically true, i.e. they assume the value T for all values of the variables.

It is easily verified directly that the axioms of the propositional calculus are of this sort.

We shall show that if a formula $\mathfrak{A}(A)$, which contains the variable A, is identically true, then the formula $\mathfrak{A}(\mathfrak{B})$ obtained from $\mathfrak{A}(A)$ by a substitution is also identically true. In fact, $\mathfrak{A}(A)$ takes on the value T for all values of the variables. In this case, $\mathfrak{A}(T)$ and $\mathfrak{A}(F)$ have the value T whatever the values of the other variables are. But for arbitrary values of the variables $\mathfrak{B}$ can have only the value T or F. It is clear from this that $\mathfrak{A}(\mathfrak{B})$ will always have the value T.

We shall prove that if the formulae $\mathfrak{A}$ and $\mathfrak{A} \rightarrow \mathfrak{B}$ are identically true, then the formula $\mathfrak{B}$ is also identically true.

If $\mathfrak{A}$ is identically true, then it always has the value T. Since the formula $\mathfrak{A} \rightarrow \mathfrak{B}$ also always assumes the value T, $\mathfrak{B}$ cannot take on the value F for any values of the variables whatsoever, for otherwise the formula $\mathfrak{A} \rightarrow \mathfrak{B}$ would assume the value $T \rightarrow F$ which is F according to the definition of implication in propositional algebra.

We have thus shown that: (1) all axioms are identically true formulae, (2) applying the rule of inference to identically true formulae we obtain formulae which are also identically true. It follows from this that all deducible formulae of the propositional calculus, considered as formulae in the propositional algebra, are identically true. It is clear in this case that if the formula $\mathfrak{A}$ is deducible in the propositional calculus, then the formula $\bar{\mathfrak{A}}$ cannot be deduced inasmuch as $\mathfrak{A}$ is an identically true formula, and then $\bar{\mathfrak{A}}$, conversely, assumes the value F for all values of the variables occurring in it. The consistency of the propositional calculus is thus proved.

§10. Completeness of the propositional calculus

We have already noted, in §8, that the formulae of the propositional calculus can be interpreted as formulae in propositional algebra. In proving the consistency of the propositional calculus we showed that every formula

which is deducible in the propositional calculus is identically true if it is considered as a formula of the propositional algebra. The converse question arises: Is every identically true formula of propositional algebra deducible in the propositional calculus.

This question is the completeness problem in the wide sense for the propositional calculus.

The meaning of such a formulation of the question consists in that, in the construction of a logical calculus intended for the expression of a formal logic, we must know whether we have sufficient axioms and rules in order to deduce an arbitrary formula which in the formal interpretation is identically true.

The completeness problem in the wide sense is solved in the affirmative.

THEOREM. *Every identically true formula of propositional algebra is deducible in the propositional calculus.*

Proof. To prove this theorem, we consider any formula A which is identically true in the propositional algebra and utilize Theorem 8, §7:

$$\vdash \prod_{\delta_1, \ldots, \delta_n = T, F} A(\delta_1, \ldots, \delta_n) \to A(A_1, \ldots, A_n). \qquad (1)$$

Since the formula $A = A(A_1, \ldots, A_n)$ is identically true, an arbitrary substitution of the formulae T, F in place of $A_1, \ldots, A_n$ in the formula A leads to a true formula, i.e. to one which is true in the propositional calculus (this is proved in §8). Consequently, in the formula

$$\prod_{\delta_1, \delta_2, \ldots, \delta_n = T, F} A(\delta_1, \ldots, \delta_n)$$

which is the antecedent in (1), all factors are true formulae and hence this entire formula is true. Now applying the rule of inference to (1) we find that the formula $A(A_1, \ldots, A_n)$ is true, i.e. it is deducible in the propositional calculus. Completeness of the propositional calculus in the wide sense is thus proved. We showed that the expression "formula true in the propositional calculus" which we used previously coincides with the formal concept of an identically true formula. One of the corollaries to the completeness theorem is the possibility of carrying over directly into the propositional calculus all "rules of operation" with the formulae which are satisfied in propositional algebra. For example, it follows from this that the following assertions are valid in the propositional calculus:

$$\vdash A \,\&\, B \sim B \,\&\, A,$$

$$\vdash A \vee B \sim B \vee A,$$

$$\vdash A \,\&\, (B \,\&\, C) \sim (A \,\&\, B) \,\&\, C,$$

$$\vdash A \vee (B \vee C) \sim (A \vee B) \vee C,$$

$$\vdash A \,\&\, (B \vee C) \sim A \,\&\, B \vee A \,\&\, C,$$

$$\vdash A \vee B \,\&\, C \sim (A \vee B) \,\&\, (A \vee C),$$
$$\vdash (A \rightarrow B) \sim \bar{A} \vee B,$$
$$\vdash \overline{A \vee B} \sim \bar{A} \,\&\, \bar{B},$$
$$\vdash \overline{A \,\&\, B} \sim \bar{A} \vee \bar{B}.$$

The concept of the completeness of a logical calculus in the narrow sense has no less important significance than the concept of completeness in the wide sense. A logical calculus is said to be complete in the narrow sense if the adjunction to it, as an axiom, of any formula which is not deducible in it leads to a contradiction.

The propositional calculus is also complete in the narrow sense. The proof of this fact is not difficult to carry out, by making use of the conjunctive normal forms. We leave the proof of this fact to the reader.

§11. Independence of the axioms of the propositional calculus

As we have already mentioned above, every logical calculus can be given in the following way: we define the concept of formula and the concept of true formula. This is done by indicating, first, certain initial formulae which are declared to be true and called axioms, and second, rules of inference, i.e. rules with the aid of which new true formulae can be formed from true formulae. For every such calculus the question arises of the independence of its axioms. This question is posed in the following form:

Can any of the axioms be deduced from the remaining ones by applying the rules of inference of the given system.

If it turns out that some axiom can be deduced in this way from the remaining, then it can be eliminated from the list of axioms and this does not modify the logical calculus, i.e. the store of its true formulae remains the same.

An axiom which is not deducible from the remaining axioms is said to be independent of the latter and a system of axioms in which no axiom is deducible from the remaining is said to be an independent system of axioms. Otherwise the system of axioms is called dependent. It is clear that a dependent system of axioms is less perfect in some sense than an independent system inasmuch as it contains superfluous axioms. It appears at first glance that the question of the independence of a system of axioms is of little importance and is of significance only from the viewpoint of technical convenience. This is however not always so. The question of the independence of an axiom of some system from the other axioms is frequently equivalent to the question of the possibility of replacing the given axiom by its negation without causing a contradiction in the system under consideration. As an example, we can point to the question of the independence of the fifth postulate of Euclid in the system of axioms of geometry. As is known, this question was of great significance in the development of mathematics.

We shall prove that the system of axioms of the propositional calculus is independent. The method of proof of this proposition is comparable to the one we used in the proof of the consistency of the propositional calculus (see §9). We then interpreted the variables of the propositional calculus as variables in propositional algebra, which are capable of assuming the two values T and F. In this connection, we defined the operations &, V, $\rightarrow$, — the same way as in propositional algebra and we established that every deducible formula in the propositional calculus takes on the value T for all values of the variables. To solve the problem of the independence of an axiom A in the propositional calculus, we pose the task of interpreting the variables of the propositional calculus as variables which are capable of taking on a finite set of values which we shall denote by the Greek letters α, β, ... We define the operations &, V, $\rightarrow$, — with the proviso that the following conditions be maintained:

(1) All axioms except the axiom A take on the value α for all values of the variables.

(2) Every formula which is deducible from the set of all axioms of the system, except A, also takes on the value α for all values of the variables which occur.

(3) The axiom A takes on values distinct from α for certain values of the variables occurring.

It is clear that if one succeeds in introducing such an interpretation, then the independence of the axiom A from the other axioms will be proved, for if A were deducible from them, then it would have the value α for all values of the variables. We note that the formulae in which the variables are replaced by their values also have meaning. For instance,

$$\alpha \mathbin{\&} \beta, \quad \bar{\alpha}, \quad A \rightarrow \alpha, \text{ and so on.}$$

The fact that two formulae A and B assume the same values α, β, ... for all replacements of the variables occurring in them will, for the sake of brevity, be expressed by the equality sign:

$$A = B.$$

In this connection we shall always assume that the sign $=$ is a weaker connective than the logical connectives &, V, $\rightarrow$.

It is easiest to prove the independence of the axioms of groups II-IV. We shall now prove the independence of axiom II.1. To this end we shall interpret the variables of the propositional calculus as variables which assume the two values α and β. In this connection, α will play the role of T and β that of F. All logical operations, except multiplication, will be defined as in propositional algebra. We write this interpretation out in detail:

$$\alpha \rightarrow \alpha = \alpha; \quad \beta \rightarrow \beta = \alpha; \quad \beta \rightarrow \alpha = \alpha; \quad \alpha \rightarrow \beta = \beta;$$
$$\alpha \vee \alpha = \alpha; \quad \alpha \vee \beta = \alpha; \quad \beta \vee \alpha = \alpha; \quad \beta \vee \beta = \beta;$$
$$\bar{\alpha} = \beta; \quad \bar{\beta} = \alpha.$$

And the multiplication operation will be defined by the condition:

$$A \& B = B.$$

We shall show that all the formulae I-IV, except II.1, take on the value α for all values of the variables occurring. Multiplication does not appear in the formulae of the groups I, III and IV; and the remaining operations are defined as in propositional algebra. Since these formulae are identically true in the propositional algebra, in our interpretation they assume the value α for all values of the variables. We consider the formulae of group II separately. Formula II.2 always takes on the value α inasmuch as in our interpretation it is equivalent to

$$B \to B.$$

Formula II.3 is equivalent to the formula

$$(A \to B) \to ((A \to C) \to (A \to C)).$$

This formula does not contain the operation of multiplication and is an identically true formula of the propositional algebra. It therefore always assumes the value α.

But formula II.1 is not identically equal to α. In fact, for $A = \beta$ and $B = \alpha$, it takes on the form:

$$\beta \& \alpha \to \beta.$$

But, by the definition of the operation &, we have

$$\beta \& \alpha = \alpha.$$

Our formula therefore takes on the form

$$\alpha \to \beta,$$

and this expression has, by agreement, the value β.

We shall now show that the formulae obtained by means of the rules of inference from formulae which are identically equal to α are themselves identically equal to α.

For the substitution rule this is obvious: if a formula assumes the constant value α for all values of the variables, then the formula obtained from it by an arbitrary replacement of the variables will be of the same sort.

We consider the rule of deduction. Suppose the formulae A and $A \to B$ take on the value α for all values of the variables occurring in them; then

$$A \to B = \alpha \to B.$$

In this case B cannot take on the value β for then we should have

$$A \to B = \alpha \to \beta = \beta,$$

which cannot transpire. (For the rule of deduction, we repeated here essentially the line of argument of §10 where it is proved that the rule of deduction applied to identically true formulae in the sense of propositional algebra lead to formulae of the same sort.)

We have thus proved the independence of axiom II.1.

In general, the independence of an arbitrary axiom from groups II-IV can be proved by the following scheme: we assume that the variables can take on only two values α and β. All the logical operations &, $\vee$, $\rightarrow$, —, except one of them, are defined as in propositional algebra where α plays the role of T and β that of F. But we define one of the operations in such a way that the axiom whose independence is being proved is not identically equal to α. Instead of carrying out all these proofs, we produce a table in the first column of which appear the axioms whose independence is being proved, in the second column appears the definition of that operation &, $\vee$, $\rightarrow$ or — which is defined differently than in the propositional algebra, and in the third column are indicated those values of the variables for which the corresponding axiom takes on the value β.

For all indicated interpretations, the formulae, occurring in the group which are different from the one in which the axiom being investigated occurs, take on the value α for all values of the variables. This transpires because that exclusive operation which is defined otherwise than in the propositional algebra does not occur in the axioms of these groups and, consequently, the interpretation of these formulae is the same as in propositional algebra. Therefore all these formulae take on the value α for all values of the variables.

Axioms	*Excluded operation*	*Values of the variables*
II.1. $A \& B \rightarrow A$	$A \& B = B$	$A = \beta,\ B = \alpha$
II.2. $A \& B \rightarrow B$	$A \& B = A$	$A = \alpha,\ B = \beta$
II.3. $(A \rightarrow B) \rightarrow$ $\rightarrow ((A \rightarrow C) \rightarrow$ $\rightarrow (A \rightarrow B \& C))$	$A \& B = \beta$	$A = \alpha,\ B = \alpha,\ C = \alpha$
III.1. $A \rightarrow A \vee B$	$A \vee B = B$	$A = \alpha,\ B = \beta$
III.2. $B \rightarrow A \vee B$	$A \vee B = A$	$A = \beta,\ B = \alpha$
III.3. $(A \rightarrow C) \rightarrow$ $\rightarrow ((B \rightarrow C) \rightarrow$ $\rightarrow (A \vee B \rightarrow C))$	$A \vee B = \alpha$	$A = \beta,\ B = \beta,\ C = \beta$
IV.1. $(A \rightarrow B) \rightarrow$ $\rightarrow (\bar{B} \rightarrow \bar{A})$	$\bar{A} = A$	$A = \beta,\ B = \alpha$
IV.2. $A \rightarrow \bar{\bar{A}}$	$\bar{A} = \beta$	$A = \alpha$
IV.3. $\bar{\bar{A}} \rightarrow A$	$\bar{A} = \alpha$	$A = \beta$

For axioms of that group in which the axiom under investigation occurs one can verify directly that two of them are also identically equal to α and the axiom being investigated itself takes on the value α for all values of the variables, indicated in the third column.

The proof of the fact that the rules of deduction applied to the formulae which are identically equal to α generate formulae which are also identically

equal to α for all interpretations which we gave above in the proof of the independence of axiom II.1 is left unchanged. It thus remains to prove the independence of the axioms of group I. The proof of the independence of these axioms is much more difficult since the sign $\rightarrow$ occurs in all groups.

Interpretations which we shall use for the proof of the independence of the axioms of group I satisfy the following general conditions:

$$\left.\begin{array}{lll}
A \rightarrow A = \alpha; & A \rightarrow \alpha = \alpha; & \beta \rightarrow A = \alpha; \\
A \,\&\, B = B \,\&\, A; & A \,\&\, \alpha = A; & A \,\&\, \beta = \beta; \\
A \vee B = B \vee A; & A \vee \alpha = \alpha; & A \vee \beta = A; \\
\bar{\alpha} = \beta; & \bar{\beta} = \alpha; & A \,\&\, A = A;\ A \vee A = A.
\end{array}\right\} \text{(a)}$$

It is clear that these conditions are compatible since they are satisfied, for instance, by the interpretation which is a propositional algebra if α is taken for T and β for F. But these conditions do not determine the interpretation uniquely, as we shall see presently.

To prove the independence of axioms I.1, we choose the following interpretation. The variables take on the values α, β, γ and δ, and, besides the condition (a), the following conditions must be satisfied:

$$\left.\begin{array}{lll}
\alpha \rightarrow \beta = \beta; & \alpha \rightarrow \gamma = \beta; & \alpha \rightarrow \delta = \beta; \\
\gamma \rightarrow \beta = \beta; & \gamma \rightarrow \delta = \beta; & \\
\delta \rightarrow \beta = \beta; & \delta \rightarrow \gamma = \alpha; & \\
\gamma \,\&\, \delta = \delta; & \gamma \vee \delta = \gamma; & \bar{\gamma} = \delta;\ \bar{\delta} = \gamma.
\end{array}\right\} \text{(b)}$$

It is easily seen that conditions (a) and (b) already determine the interpretation completely. For example, the operation $\rightarrow$ is determined by conditions (a) and (b) uniquely. In fact, when the first of the members in the implication is β or when the second is α or when both members are equal, then the operation $\rightarrow$ is determined by conditions (a). In all remaining cases, the value of the formula $A \rightarrow B$ is determined by conditions (b).

The operations & and $\vee$ are also determined by conditions (a) when one of the members either equals α or equals β or when both members are equal.

One case remains — when one of the members equals γ and the other δ; in this case, the operations are completely determined by the conditions $A \,\&\, B = B \,\&\, A$; $A \vee B = B \vee A$ from (a) and conditions (b). The negation operation is determined in an obvious manner by conditions (a) and (b).

It follows from conditions (a) and (b) that the application of the rule of inference to the formula which is identically equal to α also leads to a formula which is identically equal to α.

In fact, if $A = \alpha$ and $A \rightarrow B = \alpha$, then $\alpha \rightarrow B = \alpha$. But it is clear from conditions (a) and (b) that if the value of B is different from α, then $\alpha \rightarrow B$ is never equal to α, and therefore B is also identically equal to α.

That a substitution into a formula which is identically equal to α leads to

a formula which is identically equal to α obviously remains valid for all interpretations. Thus, the rules of inference applied to formulae which are identically equal to α lead to formulae which are also identically equal to α. Moreover, the interpretation just introduced possesses the property that if the variables assume the values α and β, then the operations &, V, $\rightarrow$ and — on them are the same as in propositional algebra if one takes α for T and β for F.

In the interpretation under consideration, formula I.1 takes on the value β for the values $A = \delta$, $B = \alpha$ of the variables. In fact, for these values, this formula assumes the form: $\delta \rightarrow (\alpha \rightarrow \delta)$, but $\alpha \rightarrow \delta = \beta$ in virtue of conditions (b), and so the formula takes on the form: $\delta \rightarrow \beta = \beta$.

It can be shown that all the remaining axioms take on the value α for all values of the variables. We shall not carry out the proof of this assertion for all the axioms. Its validity can always be established by a direct verification. We shall limit ourselves to proving the truth of our assertion for certain axioms.

We shall prove that axiom I.2 is identically equal to α. We first write it out:

$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)).$$

To begin with, we show that if

$$A \rightarrow B = \alpha; \quad B \rightarrow C = \alpha,$$

then also $A \rightarrow C = \alpha$ and, consequently, axiom I.2 assumes the value α.

In fact, if $A \rightarrow B = \alpha$, then only the following cases are possible:

$$(1)\ A = \beta; \quad (2)\ B = \alpha; \quad (3)\ A = B; \quad (4)\ A = \delta;\ B = \gamma.$$

In the first three cases it is directly evident that

$$A \rightarrow C = \alpha.$$

In the last case, in virtue of the fact that $B \rightarrow C = \alpha$, only two values are possible for C: γ and α. In both cases, $A \rightarrow C = \alpha$.

It remains to consider the case when either $A \rightarrow B$ or $B \rightarrow C$ is not equal to α. We note that the implication, as can easily be seen from (a) and (b), can assume only the values α or β. If $A \rightarrow B = \beta$, then

$$(A \rightarrow B) \rightarrow (A \rightarrow C) = \alpha$$

and the entire formula I.2 takes the value α.

It remains to consider the case when

$$B \rightarrow C = \beta.$$

If, in this connection, A takes a value different from β, then

$$A \rightarrow (B \rightarrow C) = \beta$$

and the entire formula I.2 takes the value α.

But if $A = \beta$, then $A \to C = \alpha$ and formula I.2 again takes on the value α. Thus, I.2 takes on the value α for arbitrary values of the variables.

We now verify axiom II.1:

$$A \,\&\, B \to A.$$

In the case when A or B takes the value β, we have

$$A \,\&\, B = \beta,$$

and therefore

$$A \,\&\, B \to A = \alpha.$$

But if $A = \alpha$, then axiom II.1 also takes the value α.

If $B = \alpha$, then

$$A \,\&\, B = A; \quad A \,\&\, B \to A = A \to A = \alpha,$$

and formula II.1 again takes the value α. If $A = B$, then axiom II.1 becomes equal to the formula

$$A \,\&\, A \to A.$$

But in virtue of the last and first of conditions (a), we have

$$A \,\&\, A \to A = \alpha.$$

It remains to consider the case when A and B take on the values δ and γ whereby A is not equal to B.

Taking the commutativity of the operation & into consideration, it is sufficient to consider two cases:

$$\gamma \,\&\, \delta \to \gamma \quad \text{and} \quad \gamma \,\&\, \delta \to \delta.$$

But $\gamma \,\&\, \delta = \delta$. Therefore the first expression is equal to

$$\delta \to \gamma,$$

and the second to

$$\delta \to \delta.$$

In virtue of conditions (a) and (b), both these expressions have the value α.

We consider, further, the axiom IV.1:

$$(A \to B) \to (\bar{B} \to \bar{A}).$$

The implication $A \to B$, as we have already stated above, can take only the values α and β.

If

$$A \to B = \beta,$$

then formula IV.1 takes the value α. We assume that

$$A \to B = \alpha.$$

Then only the following cases are possible:

$$(1)\ A = \beta; \quad (2)\ B = \alpha; \quad (3)\ A = B; \quad (4)\ A = \delta,\ B = \gamma.$$

It is clear from (a) and (b) that in each of these cases we have

$$\bar{B} \to \bar{A} = \alpha,$$

and, consequently, formula IV.1 also takes the value α.

We now go over to the proof of the independence of axiom I.2. To this end we choose an interpretation in which the variables take on three values: α, β, γ.

The operations are defined by conditions (a) and the supplementary conditions:

$$\alpha \to \beta = \beta; \quad \alpha \to \gamma = \gamma; \quad \gamma \to \beta = \gamma; \quad \bar{\gamma} = \gamma. \tag{c}$$

The implication and negation operations are completely defined by conditions (a) and (c).

In this case, the operations & and V are completely determined by conditions (a).

Let us consider, for instance, the product A & B. If one of the variables takes on the value α or β, then A & B is determined by conditions (a). But if both variables take on the value γ, then A & B is also equal to γ.

With this interpretation, axiom I.2,

$$(A \to (B \to C)) \to ((A \to B) \to (B \to C)),$$

for certain values of the variables takes on the value γ; the remaining axioms take on the value α for all values of the variables.

We shall show the validity of the first assertion. We set

$$A = \gamma, \quad B = \gamma, \quad C = \beta.$$

We obtain

$$(\gamma \to (\gamma \to \beta)) \to ((\gamma \to \gamma) \to (\gamma \to \beta)) = (\gamma \to \gamma) \to \\ \to (\alpha \to \gamma) = \alpha \to (\alpha \to \gamma) = \gamma.$$

The proof of the second assertion will not be carried out for all the remaining axioms, but as an example we shall verify certain of them.

Let us consider axiom I.1,

$$A \to (B \to A).$$

In the case when $A = B$, this formula takes on the value α. In case $A = \alpha$, we have $B \to A = \alpha$ and formula I.1 takes on the value α. For $A = \beta$ we also have $A \to (B \to A) = \alpha$.

It remains to consider the case $A = \gamma$. We will then have

$$\gamma \to (B \to \gamma).$$

But $B \to \gamma$ can take the values α and γ. In both cases, we have

$$\gamma \to (B \to \gamma) = \alpha.$$

We consider now axiom III.1:

$$A \to A \vee B.$$

If $A = \beta$, then this formula takes the value α.

If $A = \alpha$, then $A \vee B = \alpha$, and formula III.1 again takes the value α. We assume that $A = \gamma$. Then we shall have that

$$\gamma \to \gamma \vee B.$$

If $B = \alpha$ or $B = \gamma$, then the latter expression equals α.

If $B = \beta$, then $\gamma \vee B = \gamma$, and formula III.1 takes the value α.

We shall verify still axiom IV.1:

$$(A \rightarrow B) \rightarrow (\bar{B} \rightarrow \bar{A}).$$

In the case when A and B are equal to α or β, formula IV.1 takes the value α inasmuch as the laws of propositional algebra hold then.

If $A = B$, then formula IV.1 will equal

$$\alpha \rightarrow \alpha = \alpha.$$

We shall assume that $A = \gamma$; we obtain

$$(\gamma \rightarrow B) \rightarrow (\bar{B} \rightarrow \bar{\gamma}).$$

But $\bar{\gamma} = \gamma$; consequently, the latter expression equals

$$(\gamma \rightarrow B) \rightarrow (\bar{B} \rightarrow \bar{\gamma}).$$

If $B = \alpha$, then $\bar{B} = \beta$; $\bar{B} \rightarrow \gamma = \alpha$, and the entire expression takes on the value α.

If $B = \beta$, then

$$(\gamma \rightarrow \beta) \rightarrow (\bar{\beta} \rightarrow \gamma) = (\gamma \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma) = \gamma \rightarrow \gamma = \alpha.$$

If $B = \gamma$, and $A = \alpha$, then

$$(\alpha \rightarrow \gamma) \rightarrow (\gamma \rightarrow \alpha) = (\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma) = \gamma \rightarrow \gamma = \alpha.$$

Finally, if $B = \gamma$ and $A = \beta$, then

$$(\beta \rightarrow \gamma) \rightarrow (\bar{\gamma} \rightarrow \bar{\beta}) = (\beta \rightarrow \gamma) \rightarrow (\gamma \rightarrow \alpha) = \alpha \rightarrow \alpha = \alpha.$$

Thus, formula IV.1 takes on the value α for all values of the variables. The same thing is proved for all the remaining axioms analogously.

The independence of the system of axioms of the propositional calculus is thus proved.

NOTE. The author's use of the term *deduction* both for a proof of a formula from the axioms and for a proof with additional assumptions is unusual It is more common to use the term deduction only in the second case and to speak of proof in the first case. However this dual use is deliberate and defensible, and the author confines the term proof to metaliogical contexts. [R.L.G.]

CHAPTER III

PREDICATE LOGIC

We saw above that it is possible to give two different descriptions of propositional logic. In Chapter I, we gave its informal description, called propositional algebra, and in the second chapter we discussed it in the form of an axiomatic system.

Proceeding to the consideration of another logical system, which we shall call predicate logic, we shall first discuss it informally—in the spirit of propositional algebra and in Chapter IV we shall discuss predicate logic in the form of an axiomatic calculus.

We note, however, that whereas we did not need, for the description of propositional calculus, to go beyond constructive limits, in predicate logic the situation is different. In order to discuss it informally, we must bring in the concept of actual infinity and accept, without any foundation, methods of argument which are utilized in the theory of sets. In such a discussion of predicate logic, we cannot, of course, pose the problem of the foundations of mathematics since that which is especially in need of this foundation—the theory of sets—will be taken as the basis of our discussion.

The formal treatment of predicate logic, however, possesses the merit that it very much facilitates the study of predicate calculus as well as of other abstract logical systems. Although it is not an axiomatic system itself, it comprises rich heuristic means enabling one to orientate oneself easily in many problems touching upon axiomatic logical systems.

§1. Predicates

Predicate logic is a development of propositional algebra. It comprises all of propositional algebra, i.e. elementary propositions, considered as quantities which assume the two values T and F, all operations of propositional algebra, and, consequently, all its formulae, but brings into consideration besides this, all propositions about objects. In predicate logic we analyse propositions into subject and predicate.

Let M be a set of objects and suppose a, b, c, d are any well-defined objects from this set. Propositions about these objects will be denoted in the form

$$P(a),\ Q(b),\ R(c, d), \text{ and so forth.}$$

$P(a)$ denotes a proposition about the object a, $Q(b)$ a proposition about the object b, $R(c, d)$ a proposition about the objects c and d, and so on. Suppose, for example, that M is the set of natural numbers and that the letters a, b, c, d are the numbers 5, 8, 3, 1, respectively. Then $P(a)$ might be, for instance, the proposition "5 is a prime number", $Q(b)$ might be "8 is an odd natural number", and $R(c, d)$ might be "3 is greater than 1".

Such propositions can be true as well as false. As in propositional algebra, we shall consider these propositions only from the point of view that they represent either truth or falsity, which are denoted by the symbols T and F respectively. But, in distinction from propositional algebra, we shall here assume that the values T and F are put into correspondence with well-defined objects or groups of objects. Thus, in the examples considered above, $P(a)$ is T, put into correspondence with 5; $Q(b)$ is F, put into correspondence with 8; $R(c, d)$ is F, put into correspondence with the pair 3, 1.

Suppose M is an arbitrary non-void set and that x is an arbitrary object in this set. Then the expression $F(x)$ denotes the proposition which becomes well-defined when x is replaced by a well-defined object from M. $F(a), F(b), \ldots$ are already well-defined propositions. For example, if M is the set of natural numbers, then $F(x)$ can denote "x is a prime number".

This indeterminate proposition becomes well defined if x is replaced by some number—for example: "3 is a prime number", "4 is a prime number", and so on.

Let $S(x, y)$ denote "x is less than y".

This proposition becomes well defined if x and y are replaced by numbers: "1 is less than 3", "5 is less than 2", and so on.

Since, from our point of view, every well-defined proposition is either T or F, the expression $F(x)$ denotes that to every object in M there is made to correspond one of the two symbols T or F. In other words, $F(x)$ is a function which is defined on the set M and assumes only the two values T and F. In the same way, the indefinite propositions in two or more objects $H(x, y)$, $G(x, y, z)$, and so on, are functions of two, three, and so forth, variables. In this connection, the variables x, y, z range over the set M and the functions take on the values T and F. These indefinite propositions, or functions of one or several variables, will be called *logical functions* or *predicates*. By means of a predicate with one variable, one can express a *property of an object*—for instance, "x is a prime number", "x is a right triangle", and so forth.

The concept of predicate in the classical logic of Aristotle corresponds in our terminology to a predicate with one variable. The concept of predicate which we have introduced has a more extended meaning. We shall also call logical functions of several variables predicates. One can express *relations* among objects by means of such predicates. Suppose, for example, that M is the set of real numbers and that the variables $x, y, z, \ldots$ are objects in M.

Then by means of predicates of two or a greater number of variables, we can express the different relations among numbers, such as:

$$x < y; \quad x + y + z = 0;$$

and others, denoting these predicates respectively by $A(x, y)$, $B(x, y, z)$, and so forth. Let M be the set of members of a family. Then by means of predicates one can express familial relations, for example "to be father and son", "to be brother and sister", and so on. The predicate $L(x, y)$ can denote "x is the father of y", $M(x, y)$ can denote "x and y are brothers", and so on.

We shall see later that the introduction of predicates of several variables which are capable of expressing relations among objects brings in something essentially new in comparison with the predicate logic of one variable. It turns out that in all systems of axioms of mathematical disciplines there exist axioms which are not expressible in terms of predicates of one variable. But all axioms of these systems can be expressed in terms of predicates of a greater number of variables.

All concepts which we shall introduce always refer to some arbitrary set M which we shall call a *field*. The elements of this field will be denoted by lower-case Latin letters (and sometimes we shall allow these letters indices). Letters at the end of the Latin alphabet—

$$x, y, z, u, v, x_1, x_2, \ldots$$

—denote indefinite elements of the field. We shall call them *object variables*. Letters at the beginning of the alphabet—

$$a, b, c, a_1, a_2, \ldots$$

—denote well-defined objects of the field. We shall call them *individual objects* or *object constants*.

As in propositional algebra, we shall denote by upper-case Latin letters

$$A, B, \ldots, X, A_1, A_2, \ldots$$

variables which take on the values T and F. We shall call them *propositional variables*. We shall also consider constant propositions, denoting them by upper-case Latin letters which will be specified in some way or simply with some supplementary agreement.

The expressions

$$F(x), G(x, y), P(x_1, \ldots, x_n), A(x, x), \ldots$$

denote predicates, i.e. functions whose arguments take on values from the field M, and the functions themselves can take on only two values: T and F. If such an expression is not specified in any way and no agreements are made, then it denotes a variable (i.e. an arbitrary) predicate on the field M of the prescribed number of variables. And a definite predicate will be denoted by the same symbol with the corresponding agreement or some supplementary

remark. Moreover, certain predicates will be denoted by those symbols which are generally used for them.

For example, the predicate "x is less than y" will be denoted by $x < y$, the predicate "x equals y" will be denoted by $x = y$, and so forth.

Propositions, which are expressed by upper-case Latin letters—variables as well as constants—and also the expressions

$$F(a),\ G(a, b),\ \ldots,$$

where F, G are predicates and a, b are individual objects, will be called *elementary propositions.*

Upper-case Latin letters and predicate symbols, of individual objects as well as object variables, will be called elementary formulae. This phrase will be used in order to distinguish these formulae from compound formulae which will be constructed from elementary ones.

The symbols for objects are not formulae. Elementary formulae—propositions as well as logical functions—are always quantities which are capable of taking on only the values T and F. Therefore, elementary formulae can be combined by means of the operations of propositional algebra:

$$\&,\ \vee,\ \rightarrow,\ -,$$

retaining for these operations that definition which we gave them in propositional algebra. The formulae thus obtained can, in their turn, define propositions or predicates. For example:

(1) $A \vee F(x)$,

(2) $A(x, y) \rightarrow (B \,\&\, \bar{A}(x, x))$,

(3) $G(x, y) \rightarrow G(x, x)$,

(4) $L(x) \sim L(y)$, and so forth.

The first formula, for fixed A and $F(x)$, defines some predicate. The fourth formula represents a predicate of the two variables x and y for every L. If x and y take on the same values, then this predicate takes on the value T.

§2. Quantifiers

Besides the operations of propositional algebra, we shall make use of two new operations. We did not have these operations in propositional algebra, inasmuch as they are related to the peculiarities of predicate logic. These operations express the assertions of universality and existence.

1. The universal quantifier. Suppose $R(x)$ is a completely definite predicate which takes on the values T or F for every element x of some field M. Then the expression

$$(x)R(x)$$

will be understood to be *a true proposition when $R(x)$ is true for every element*

x of the field M and false otherwise. This proposition no longer depends on x. The verbal expression corresponding to it will be: "$R(x)$ is true for every x".

Suppose now that $A(x)$ is a formula in predicate logic, which takes a definite value if the object variables and predicate variables occurring in it are replaced in a completely definite manner. The formula $A(x)$ can also contain other variables besides x. Then the expression $A(x)$, upon the replacement of all variables—object as well as predicate variables—except x, represents a concrete predicate which depends only on x. And the formula $(x)\, A\, (x)$ becomes a completely defined proposition. Consequently, this formula is completely determined by giving the values of all variables, except x, and, hence it does not depend on x. The symbol (x) is called the *universal* quantifier.

2. The existential quantifier. Let $R(x)$ be a predicate. We relate the formula $(\exists x)R(x)$ to it by defining its value to be truth if there exists an element in the field M for which $R(x)$ is true and falsity otherwise. Then if $A(x)$ is a defined formula in predicate logic, the formula $(\exists x)A(x)$ is also defined and does not depend on the value of x. The symbol $(\exists x)$ is called the *existential* quantifier.

The quantifiers $(\exists x)$ and (x) are called the *duals* of each other.

We shall say that the quantifiers (x) and $(\exists x)$ in the formulae $(x)A(x)$ and $(\exists x)A(x)$ apply to the variable x or that the variable x is associated with the corresponding quantifier.

An object variable which is not associated with any quantifier will be called a *free variable.* We have thus described all formulae of predicate logic.

If two formulae A and B associated with some field M assume the same value—T or F—for all substitutions of the predicate variables, propositional variables, and free object variables by individual predicates defined on M, by individual propositions and individual objects from M respectively, then we shall say that these formulae are equivalent on the field M. (In the substitutions of the predicate, propositional and object variables those of them which are denoted in the formulae A and B in the same way must of course be substituted also in the same way.)

If two formulae are equivalent on arbitrary fields M, then we shall simply call them equivalent. As in propositional algebra, equivalent formulae can be substituted for one another.

The equivalence of formulae allows one to reduce them, as occasion arises, to a more suitable form.

Clearly, all equivalences which hold in propositional algebra also carry over to predicate logic. In particular,

$$A \rightarrow B \text{ is equivalent to } \bar{A} \vee B$$

holds.

Utilizing this fact for any formula, we can find a formula equivalent to

t in which, of the operations of propositional algebra, we have only &, V and —.

EXAMPLES:

1. $(\exists x)(A(x) \rightarrow (y)B(y))$ is equivalent to $(\exists x)(\bar{A}(x) \vee (y)B(y))$.
2. $(x)A(x) \rightarrow (B(z) \rightarrow (x)C(x))$ is equivalent to $\overline{(x)A(x)} \vee (\bar{B}(z) \vee (x)C(x))$.
3. $((\exists x)A(x) \rightarrow (y)B(y)) \rightarrow C(z)$ is equivalent to

$$\overline{(\exists x)A(x) \rightarrow (y)B(y)} \vee C(z).$$

The latter is equivalent to

$$\overline{\overline{(\exists x)A(x)} \vee (y)B(y)} \vee C(z).$$

Performing transformations of propositional algebra, we obtain the equivalent formula:

$$(\exists x)A(x) \,\&\, \overline{(y)B(y)} \vee C(z).$$

In predicate logic, we have—besides the equivalences of propositional algebra—equivalences associated with quantifiers.

There exists a law which connects quantifiers with the law of negation. Let us consider the expression

$$\overline{(x)A(x)}.$$

The proposition "$(x)A(x)$ is false" is equivalent to the proposition "there exists an element y for which $A(y)$ is false", or what amounts to the same thing "there exists an element y for which $\bar{A}(y)$ is true". Consequently, the expression $\overline{(x)A(x)}$ is equivalent to the expression $(\exists y)\bar{A}(y)$.

We consider the expression

$$\overline{(\exists x)A(x)}$$

in the same way. This is the proposition "$(\exists x)A(x)$ is false". But such a proposition is equivalent to the expression: "for all y, $A(y)$ is false" or "for all y, $\bar{A}(y)$ is true". Thus, $\overline{(\exists x)A(x)}$ is equivalent to the expression

$$(y)\bar{A}(y).$$

We have thus obtained the following rule:

If the negation symbol is moved to the right of the quantifier symbol the quantifier must be replaced by its dual.

We have already seen that for every formula there exists a formula equivalent to it which, of the propositional algebra operations, contains only &, V and —.

Utilizing the last equivalences, which are associated with the quantifiers, and the laws of propositional algebra, for every formula we can find an equivalent one in which the negation symbols apply only to elementary propositions and elementary predicates. The proof of this assertion will not be carried out here—we shall limit ourselves to an example only.

EXAMPLE. Let us consider the formula

$$\overline{(\exists x)(A(x) \rightarrow (y)B(y))}.$$

We first find an equivalent formula which does not contain the symbol $\rightarrow$. This will be

$$\overline{(\exists x)(\bar{A}(x) \vee (y)B(y))}.$$

Applying the rule considered above to the negation over the quantifier $(\exists x)$, we obtain the equivalent formula:

$$(x)\overline{(\bar{A}(x) \vee (y)B(y))}.$$

Next, performing transformations from propositional algebra, we obtain

$$(x)(A(x) \mathbin{\&} \overline{(y)B(y)}).$$

Applying again the rule for transferring the negation symbol to the right of the quantifier (y), we finally obtain

$$(x)(A(x) \mathbin{\&} (\exists y)\bar{B}(y)).$$

In this formula the negation symbol applies to the elementary predicate $B(y)$.

Formulae in which of the operations of the propositional algebra only the operations &, $\vee$ *and* — *occur and the negation signs apply only to elementary predicates and propositions will be called reduced formulae.*

From what we stated above, we can conclude that *for every formula there exists a reduced formula which is equivalent to it.* This reduced formula will be called *the reduced formula* of the given formula.

§3. The set-theoretic interpretation of predicates

The situation that two logical quantities—be they propositions or predicates—always assume the same values T or F will be denoted by means of the symbol $\equiv$. In the present section we shall consider certain sets which correspond to logical expressions. We shall use the symbol $=$ to denote the identity of two sets.

Let M be a set on which predicates are defined. We shall also call such a set a *field.* To every predicate of one variable $F(x)$ one can correlate the set of those elements a of the field M for which $F(a)$ is true. We denote this set by E_F. Conversely, to every set E contained in M one can correlate the predicate $P(x)$ which represents the proposition $x \in E$. The predicate $P(x)$ assumes the value T on E and F outside of E. Consequently E is E_P. The correspondence between subsets of M and predicates of one variable defined on it is thus one-to-one. In the sequel, if we do not explicitly state otherwise, we shall assume that the field M is non-void.

As is known, the set consisting of all elements in the set E_1 and all elements of the set E_2 is called the set-theoretic *sum* $E_1 \cup E_2$ of the two sets E_1 and E_2. The set of all elements belonging simultaneously to both the set E_1 and to the set E_2 is called the *product* or *intersection* $E_1 \cap E_2$ of the two

sets E_1 and E_2. The set-theoretic sum and set-theoretic product of an arbitrary finite or infinite number of sets are defined analogously.

Let

$$P(x) \equiv P_1(x) \vee P_2(x).$$

Then

$$E_P = E_{P_1} \cup E_{P_2},$$

i.e. E_P is the set-theoretic sum of E_{P_1} and E_{P_2}. In fact, if $x \in E_P$, then $P(x)$ is true; this means that $P_1(x)$ or $P_2(x)$ is true. In the first case $x \in E_{P_1}$ and in the second case $x \in E_{P_2}$; consequently,

$$x \in E_{P_1} \cup E_{P_2}.$$

Conversely, let $x \in E_{P_1} \cup E_{P_2}$. Then $x \in E_{P_1}$ or $x \in E_{P_2}$, i.e. $P_1(x)$ is true or $P_2(x)$ is true. Consequently, $P(x)$ is true and $x \in E_P$.

It can be shown in an analogous manner that if

$$P_1(x) \equiv P_1(x) \;\&\; P_2(x),$$

then

$$E_P = E_{P_1} \cap E_{P_2},$$

where $E_{P_1} \cap E_{P_2}$ is the set-theoretic product.

The set corresponding to the predicate $\bar{P}(x)$ is the complement to the set corresponding to the predicate $P(x)$. In set-theoretic symbols one can write that

$$E_{\bar{P}} = \mathscr{C}E_P,$$

where $\mathscr{C}E_P$ is the set of elements of the field M which do not belong to E_P or, as we say, the complement of the set E_P.

Formulae of propositional algebra, expressed by upper-case Latin letters, occurring in predicate logic can be regarded as predicates which retain the same value T or F for all objects. In virtue of our agreement, we ought to correlate the entire field M to such a predicate in the first case and the void set in the second case.

Logical laws which are valid for propositional algebra also remain valid for logical functions inasmuch as the values of these functions are these same values. In virtue of the correspondence between the logical functions and sets, to the laws of the predicate logic there correspond known laws for set-theoretic operations.

For example, to the two distributive laws for logical operations there correspond distributive laws for set-theoretic addition and multiplication. To the first distributive law,

$$F(x)[G(x) \vee H(x)] = F(x)G(x) \vee F(x)H(x),$$

there corresponds the set-theoretic law

$$P \cap (Q \cup S) = (P \cap Q) \cup (P \cap S)$$

where P, Q, S are arbitrary sets.

To the second distributive law,

$$F(x) \vee G(x)H(x) \equiv (F(x) \vee G(x))(F(x) \vee H(x)),$$

there corresponds the set-theoretic law

$$P \cup (Q \cap S) = (P \cup Q) \cap (P \cup S).$$

We have established the connection between sets and predicates of one variable. This can also be done for logical functions of a greater number of variables in an analogous manner. We shall consider only the case of functions of two variables. Let M^2 be the set of all pairs (x, y) of the set M. In this connection, we assume that pairs are distinguished not only by the occurrence of elements but also by their order.

With the function $P(x, y)$ we set into correspondence the set of those pairs (x, y) belonging to M^2 for which $P(x, y)$ is true; we denote this set by E_P as before. The connection between the functions $P(x, y)$ and the subsets of M^2 is the same as that for the case of functions of one variable and subsets of M.

We now consider the set-theoretic interpretation of quantifiers. Let

$$F(x) \equiv (\exists y)P(x, y).$$

The set E_F corresponding to the predicate F consists of those, and only those, elements of the field M for which $F(x)$, i.e. $(\exists y)P(x, y)$, is true. But the last expression is true for a given x_0 if there exists a y such that $P(x_0, y)$ is true. To the function $P(x, y)$ there corresponds the subset E_P of the set M^2. Thus, E_F consists of all those elements x of the field M for each of which there exists a pair (x, y) belonging to E_P. We call x_0 the *projection of an arbitrary pair* (x_0, y); the set of projections of all pairs belonging to a set will be called the *projection of this set.* It is easily seen that E_F is the *projection of* E_P. Let M be the set of real numbers. Similar to what is done in analytic geometry, we shall represent M^2 as a plane whose points have the co-ordinates x, y and M as the axis OX of this plane. In this case, the point x is the projection of the point (x, y) in the literal geometric sense. Therefore, the set E_F which corresponds to the logical function $F(x) \equiv (\exists y)P(x, y)$ coincides with the ordinary orthogonal projection of the set E_P on the axis OX. Denoting the projection of any set H on M by the symbol $\mathrm{pr}_x H$, we can write

$$E_F = \mathrm{pr}_x E_P.$$

In order to establish the set-theoretic meaning of the universal quantifier, we shall apply the law for the operation of negation on the operator. Let

$$F(x) \equiv (y)P(x, y).$$

Then

$$(y)P(x, y) \equiv \overline{(\exists y)\bar{P}(x, y)}.$$

As is known, one associates the set-theoretic operation of complementation with the operation of negation. Thus,

$$E_F = \mathscr{C}\mathrm{pr}_x \mathscr{C} E_P,$$

i.e. the set which corresponds to the function $(y)P(x, y)$ is the complement of the projection on M of the complement of E_P.

The converse proposition is also valid: every set R which is the projection on M of the set U and belongs to M^2:

$$R = \mathrm{pr}_x U,$$

can be represented as E_F, where $F(x)$ is $(\exists y)P(x, y)$ for $U = E_P$. In fact, to the set R there corresponds the predicate $F(x)$, defined on M, and to the set U the predicate $P(x, y)$, defined on M^2, and, obviously,

$$F(x) \equiv (\exists y)P(x, y).$$

But if R' is the complement to the projection U, then R' obviously corresponds to the predicate $(y)\bar{P}(x, y)$. In fact, to the predicate $(y)\bar{P}(x, y)$ there corresponds the set $\mathscr{C}\mathrm{pr}_x \mathscr{C} E_{\bar{P}}$, but $\mathscr{C} E_{\bar{P}} = E_P = U$ and consequently

$$\mathscr{C}\mathrm{pr}_x \mathscr{C} E_{\bar{P}} = \mathscr{C}\mathrm{pr}_x U.$$

Thus, the quantifiers are connected with the geometric operation of projection and, conversely, projection has the indicated geometric meaning.

§4. Axioms

We now consider individual predicates which we shall denote by $S(x)$ and $x = y$. The first is a function which assumes the value T for every element of the field and the second assumes the value T if x and y represent the same element, and F when x and y are distinct. The predicate $S(x)$ can be defined explicitly by means of a formula in predicate logic, for example, in the form

$$F(x) \vee \bar{F}(x),$$

where $F(x)$ is an arbitrary predicate in the same field. In fact, this expression has the value T for every x. It is impossible to represent the predicate $x = y$ directly in the form of a formula of predicate logic. But by means of such formulae one can state conditions which uniquely define the identity predicate.

We shall assume that we do not know exactly what predicate is depicted by the symbol $x = y$. We write two formulae:

1. $x = x$,
2. $x = y \rightarrow (A(x) \rightarrow A(y))$

and require that for the predicate $x = y$ these formulae be true for every predicate A and for all x, y. It is easily seen that under these conditions the predicate $x = y$ can only be an identity in x and y. In fact, if x and y are replaced by the same object, then the predicate $x = y$ assumes the value T

by virtue of formula 1. We assume that x and y are replaced by distinct objects a and b. We replace the predicate $A(t)$ by a predicate which takes on the value T if t is a and the value F if t does not coincide with a. We denote this predicate by $A^0(t)$. The formula

$$A^0(a) \rightarrow A^0(b)$$

has the value F inasmuch as $A^0(a)$ is T and $A^0(b)$ is F. But the formula

$$a = b \rightarrow (A^0(a) \rightarrow A^0(b))$$

must be true. Therefore the formula $a = b$ is false. We have thus shown that the predicate $x = y$, satisfying our conditions, can only be the predicate of identity.

One can also characterize other individual predicates in an analogous manner. Frequently, such a characterization does not define the characterizing predicate uniquely, but, rather, isolates some class of predicates. But in this case also we will call the symbol of the characterizing predicate an individual predicate.

In certain cases formulae characterize not only the individual predicate but also the field itself. This is the case when there does not exist an individual predicate, satisfying these formulae, for every field. Finally, formulae may contain no individual objects whatsoever; then they characterize the field only. For example, the formula

$$A(x) \rightarrow A(y),$$

where A is a variable predicate, characterizes the fields consisting of one element. In fact, if the field M contains only one object a, then for any replacement of x and y by elements of the field we obtain the formula

$$A(a) \rightarrow A(a)$$

which is always true. Conversely, if the field contains more than one object, then it is possible to choose a predicate for which our formula is false.

Example. We shall write down formulae characterizing a predicate which we denote by $x < y$:

1. $\overline{x < x}$,

2. $x < y \rightarrow (y < z \rightarrow x < z)$.

A predicate, depicted by the symbol $x < y$, ought to be such that formulae 1 and 2 are true for it for all values of the free variables x, y, z occurring in it, or, in other words, the predicate $x < y$ ought to satisfy conditions 1 and 2. It is quite easy to give an example of such a field and such a predicate for which our formulae are true. We consider the field of three objects a, b, c. We define a predicate for this field in the following way.

Let $a < b$ have the value T, $b < c$ the value T and $a < c$ the value T. The predicate has the value F for all the other replacements of x and y. Then

formulae 1 and 2 are satisfied for the given field—which fact is verified directly, as can be done thanks to the finiteness of the field.

In set theory, every relation $x < y$, satisfying formulae 1 and 2, is called an *order relation.* For elements which appear in the order relation $x < y$ we sometimes use the expression "x precedes y". We shall also use this expression. We say that a set is *ordered by the relation* $x < y$ if this relation satisfies, besides formulae 1 and 2, still one more formula:

$$3.\ \overline{x = y} \rightarrow (x < y \vee y < x).$$

If we understand the predicate $x = y$ informally, then formulae 1, 2, 3 are sufficient for the description of an ordered set. Otherwise, it is necessary to adjoin further formulae—characterizing equality—to the formulae 1, 2, 3.

We introduce one more concept from set theory which will be used in the sequel. *An ordered set is called well ordered if each of its non-void subsets contains an element which precedes all other elements of this subset.* In the theory of sets, one proves "Zermelo's Theorem": *Every set can be well ordered by means of some order relation.* It follows from this that for every field there exists a predicate satisfying axioms 1, 2 and 3.

Formulae which characterize individual predicates, fields, or something else, will be called axioms. Here, however, one must introduce a precise definition.

Let $A_1, A_2, \ldots, A_n$ be formulae in predicate logic which contain the predicate symbols $P_1, P_2, \ldots, P_k$ and $A_1, A_2, \ldots, A_s$ and the object symbols $a_1, a_2, \ldots, a_p$ and $x_1, x_2, \ldots, x_q$.

Suppose there exist objects $a_1^0, \ldots, a_p^0$ in the field M and predicates $P_1^0, \ldots, P_k^0$ defined on M such that after substituting them for $a_1, \ldots, a_p$ and *$P_1, \ldots, P_k$ in all formulae A_i these formulae are true for all values of the free predicate variables $x_1, \ldots, x_q$ of the field M and all possible replacements of the symbols $A_1, \ldots, A_s$ by predicates or propositions defined on the field M. We shall then say that the field M and the system of predicates $P_1^0, \ldots, P_k^0$ satisfy the system of axioms $A_1, \ldots, A_n$.*

The symbols $P_1, P_2, \ldots, P_k$ in these axioms will be called *symbols of the individual* (or *constant*) *predicates* and the symbols $A_1, A_2, \ldots, A_s$ the *symbols of the predicate variables.*

If any axiom $A(x, y, \ldots)$ contains free variables $x, y, \ldots$, then it can be replaced by another axiom:

$$(x)(y) \ldots A(x, y, \ldots).$$

In this connection, the field M and the set of individual predicates P_i^0, which satisfy the initial system of axioms, satisfy the new system also, for if the predicates $P_1^0, \ldots, P_k^0$ satisfy a system of axioms, then these axioms must be true for all values of the free variables.

A field M with predicates $P_1^0, \ldots, P_k^0$ which satisfy the system of axioms $A_1, \ldots, A_n$ is called an interpretation of this system of axioms.

§5. The consistency and independence of axioms

A system of axioms for which there exists an interpretation will be called an interpretable or formally consistent system. A system of axioms which does not admit any interpretation will be called a non-interpretable or formally inconsistent system. The definition of formal consistency or of interpretability presupposes that there exists a domain of objects from which one can form sets (fields) and define predicates on them so that from among these sets and predicates we can find interpretations for the systems of axioms that we are investigating.

The range of objects utilized in mathematics for the indicated purposes, as we have already stated in the Introduction, can consist of the set of natural numbers and everything that can be constructed from it by means of general set-theoretical constructions—in particular, the rational numbers, the real numbers, the complex numbers, various sorts of functions, and other entities.

Another sense of consistency is also possible. *We shall consider a system of axioms to be consistent if, having obtained from it any logical consequences whatsoever, we never arrive at a contradiction in the sense that we never deduce the truth and the falsity of the same assertion simultaneously.*

In order that it be possible to judge consistency in the latter sense, it is necessary to have a description of those logical methods which we require for the deduction of consequences from the axioms. We shall give the description of the logical deductions in the following chapter; it is accomplished by means of constructing an abstract logical system. After this is done, the definition of consistency in the second sense will be completely precise. In order to distinguish the different concepts of consistency introduced here we shall use the expression "*intrinsic consistency*" for consistency in the second sense. An example of intrinsic consistency of a logical system is the propositional calculus considered in the preceding chapter. Since it is impossible for us for the time being to judge the concepts introduced sufficiently rigorously and accurately, we can none the less compare these definitions of consistency to a certain degree. If one takes into consideration the fact that the domain of set-theoretic concepts, from which we draw the interpretations for systems of axioms, is itself intrinsically consistent, then it becomes clear that every interpretable system of axioms is also intrinsically consistent.

Thus, the presence of an interpretation of a system of axioms reduces the problem of the consistency of this system to the consistency of the system of concepts utilized for an interpretation. If we are convinced that this system of concepts is intrinsically consistent, then the fact that an interpretation is present establishes the intrinsic consistency of the system of axioms under investigation. The converse problem, whether an intrinsically consistent system is also interpretable, is not at all as obvious, and to solve it it is essentially

necessary to have an axiomatic description of the methods of deduction of logical consequences from the axioms. We shall not touch upon this problem here.

We now consider an arbitrary system of axioms

$$A_1, A_2, \ldots, A_n. \tag{1}$$

The axiom A_i is said to be independent of the remaining axioms if there exists a field M with predicates F_j satisfying the system of axioms

$$A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n,$$

but which does not satisfy the axiom system

$$A_1, \ldots, A_i, \ldots, A_n$$

under consideration.

The expression *the independence of one axiom of the remaining axioms*—like the expression *consistency*—is used in still another sense. We shall say that *the axiom A_i is intrinsically independent of the remaining axioms if it cannot be deduced from the remaining axioms.*

As in the case of consistency, the concept of intrinsic independence will be precise only when we give a description of the method of logical deduction of consequences from the axioms. Using non-rigorous considerations, we can compare the two definitions of independence here.

Let us assume that the axiom A_i is independent of the remaining ones in the first sense. Then there exists an interpretation for the system of axioms

$$A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \tag{2}$$

which does not satisfy the system of all the axioms together with the axiom A_i. In this case, the formula A_i cannot be logically deduced from the remaining axioms. For if it were deducible from the remaining axioms, then this deduction would also be valid for an arbitrary interpretation; for an arbitrary field with arbitrary predicates, it would follow from the truth of the axioms $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ that the axiom A_i is also true. But, by assumption, there exists an interpretation for which axiom (2) is true and A_i is not. We can conclude from this that if any axiom is independent of the remaining ones in the first sense, then it must be independent in the second sense also.

The converse problem whether an axiom which is intrinsically independent of the remaining axioms also be independent in the first sense will not be considered here.

If an axiom is not independent of the remaining axioms of the system, then we shall say that it *depends* on them. It is useful for us, however, to have a direct definition of a dependent axiom.

An axiom A_i is dependent on the remaining axioms

$$A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$$

if an arbitrary interpretation of this system satisfies the system with axiom A_i also.

The concepts of consistency and independence of a system of axioms are of great significance for mathematics. If we make use of some system of axioms, then confidence in its intrinsic consistency is absolutely necessary inasmuch as in an inconsistent system—as we have already stated earlier—there is no distinction between truth and falsity. In such a system one can prove the truth of any assertion.

Intrinsic independence is needed in order that there be no superfluous axioms in the system (we also spoke about this point earlier). This independence, as we have already seen, can also be established by means of an interpretation. The well-known history of the problem of Euclid's fifth postulate or the "parallel axiom" was connected with the problem of independence. After numerous unsuccessful attempts to prove this postulate, i.e. to deduce it from other principles of geometry, Lobachevsky expressed the opinion that this postulate cannot be deduced from the other axioms of geometry and put this assumption on a convincing foundation. The elements of the method of interpretation are already contained in his investigations and subsequently the non-deducibility of the fifth postulate was in fact finally established by this means. A system of objects was constructed which satisfies all the axioms of geometry except the axiom of parallels and which does not satisfy the latter. The method of interpretation, however, is applicable to problems of consistency and independence within known limits only. Other methods are connected with considerations of abstract logical systems. We shall not touch upon them in this chapter.

§6. One-to-one correspondences between fields

We now introduce a concept from set theory which we shall need in the sequel.

We consider two sets M and M'.

A *one-to-one correspondence* is said to be established between the elements of these sets if to each element of the set M there is assigned in some way an element of the set M'—in such a manner that to each element of M there corresponds one and only one element of M' and, conversely, to each element of M' there corresponds one and only one element of M. A one-to-one correspondence between elements of sets will be written in the form $x — x'$.

We now assume that a one-to-one correspondence between the elements of two sets M and M' has been established. Let $F(x_1, \ldots, x_n)$ be an arbitrary predicate of n variables defined on M. We define a predicate $F'(x_1', \ldots, x_n')$ on M' in the following manner: let $a_1', a_2', \ldots, a_n'$ be an arbitrary set of values of the variables $x_1', \ldots, x_n'$. To each a_i' there corresponds a definite element of the set M which we shall denote by a_i. The predicate $F(x_1, \ldots, x_n)$ is defined for all values of the variables—therefore $F(a_1, \ldots, a_n)$ has a well-

defined value, T or F. We assign precisely this value to the predicate $F'(x_1', \ldots, x_n')$ when x_1' is $a_1', \ldots, x_n'$ is a_n'. In other words, $F(a_1, \ldots, a_n)$ and $F'(a_1', \ldots, a_n')$ are both true or both false simultaneously.

The predicate $F'(x_1', \ldots, x_n')$ is thus defined on the field M'. We set this predicate into correspondence with the predicate $F(x_1, \ldots, x_n)$. It follows from our definition that also conversely to an arbitrary predicate on M' there corresponds a unique predicate defined on M.

The one-to-oneness of the above correspondence is obvious for if two predicates $F_1(x_1, \ldots, x_n)$ and $F_2(x_1, \ldots, x_n)$ defined on M are distinct, then a set of values of the variables $a_1, \ldots, a_n$ can be found for which one of the predicates assumes the value T and the other the value F.

Suppose, for instance, that $F_1(a_1, \ldots, a_n)$ has the value T and that $F_2(a_1, \ldots, a_n)$ has the value F. Then if F_1' and F_2' are predicates defined on M' corresponding to F_1 and F_2, respectively, then $F_1'(a_1', \ldots, a_n')$ has the value T and $F_2'(a_1', \ldots, a_n')$ has the value F. Therefore the predicates F_1' and F_2' are also distinct.

The correspondence between predicates which we have just established will be expressed in the same way as a correspondence between objects:

$$F - F'.$$

Let

$$x_1 - x_1', x_2 - x_2', \ldots, x_n - x_n'$$

and

$$F - F'.$$

Then

$$F(x_1, \ldots, x_n) \equiv F'(x_1', \ldots, x_n').$$

By virtue of the one-to-oneness of the correspondence, the converse also holds: if

$$F(x_1, \ldots, x_n) \equiv F'(x_1', \ldots, x_n'),$$

then

$$F - F'.$$

Suppose that A is a formula formed from certain individual predicates defined on the field M. Under every replacement of the free object variables by objects from the field M, this formula takes on a definite value, T or F. If all the predicates occurring in A are replaced by the corresponding predicates defined on M', then we obtain a formula A', referred to the field M'. It is not difficult to see that

$$A' \equiv A \tag{1}$$

for corresponding values of the variables. If the formula is constructed with the aid of the operations of propositional algebra only, then after a replacement of the predicates on M by the corresponding predicates on M' and the values of the object variables from M by the corresponding values of M', then the values of the predicates occurring in the formula remain unchanged. It is clear that after such a replacement the operations of propositional

algebra will yield the same result as before the replacement. It is furthermore easy to see that if under the indicated replacements the formula $A(x_1, \ldots, x_n)$ goes over into $A'(x_1', \ldots, x_n')$, where

$$A(x_1, \ldots, x_n) \equiv A'(x_1', \ldots, x_n'),$$

then

$$(x_i)A(x_1, \ldots, x_n) \equiv (x_i')A'(x_1', \ldots, x_n'),$$

$$(\exists x_i)A(x_1, \ldots, x_n) \equiv (\exists x_i')A'(x_1', \ldots, x_n').$$

Consequently, the operations of propositional algebra as well as the operations of quantification, performed on formulae possessing the indicated property, lead to formulae possessing the same property. It follows from what we have said that every formula composed of predicates defined on the field M also possesses this property, i.e. relation (1) holds for it.

§7. The isomorphism of fields and the completeness of systems of axioms

We now consider a field M and a system of predicates $F_1(x_1, \ldots, x_{n_1})$, $F_2(x_1, \ldots, x_{n_2})$, $\ldots$, $F_k(x_1, \ldots, x_{n_k})$ defined on this field and another field M' with the predicates $F_1'(x_1', \ldots, x_{n_1}')$, $F_2'(x_1', \ldots, x_{n_2}')$, $\ldots$, $F_k'(x_1', \ldots, x_{n_k}')$.

We shall say that the field M with the predicates F_i is isomorphic to the field M' with the predicates F_i' if a one-to-one correspondence can be established between the elements of M and M', $x - x'$, such that $F_i'(x_1', \ldots, x_n')$ has the same value (T or F) as $F_i(x_1, \ldots, x_n)$ if x_1' corresponds to x_1, $x_2' - x_2$, $\ldots$, $x_n' - x_n$.

We shall call this correspondence between the elements of the sets M and M' an *isomorphism which leaves the predicates $F_1(\ldots)$, $F_2(\ldots)$, $\ldots$, $F_k(\ldots)$ invariant.*

EXAMPLE. Suppose the field M is the set of natural numbers 1, 2, 3, 4, 5 and let one predicate $F(x, y)$ be defined on this field by the condition: the difference of the numbers x and y is exactly divisible by three, or, in other words, following the terminology of number theory, "x is congruent to y modulo three". This expression is written in the following manner:

$$x \equiv y \pmod 3.$$

More specifically, $F(x, y)$ assumes the value T if x is congruent to y modulo three and the value F otherwise.

The field M' will be the set of numbers

$$21, 22, 23, 24, 25.$$

A predicate $F'(x', y')$ for M' is defined as for M: $F'(x', y')$ is true if and only if

$$x' \equiv y' \pmod 3.$$

It is easily seen that the field M with predicate $F(x, y)$ is isomorphic to the field M' with predicate $F'(x', y')$.

We establish the following one-to-one correspondence between M and

M': $1 — 21$, $2 — 22$, $\ldots$, $5 — 25$. The differences between elements of the field M and between the corresponding elements of M' are equal and, therefore, if $x \equiv y \pmod 3$, then we also have $x' \equiv y' \pmod 3$, and conversely.

It is easily seen that if the field M with predicates $F_1, \ldots, F_k$ is isomorphic to the field M' with the predicates $F_1', \ldots, F_k'$, and the field M' in turn is isomorphic to the field M'' with the predicates $F_1'', \ldots, F_k''$, then the fields M and M'' are also isomorphic. In other words, *the relation of the isomorphism of fields is transitive.*

We consider an arbitrary system of axioms which contains the individual predicates $F_1, F_2, \ldots, F_k$ and these only. (This condition does not apply to variable predicates: they can occur in the axioms in any way.) We assume that the field M with the predicates

$$F_1^0, F_2^0, \ldots, F_k^0$$

satisfies our system of axioms. Let M' be an arbitrary field with the predicates

$$F_1', F_2', \ldots, F_k'$$

which is isomorphic to the field M. Clearly, the field M' with the predicates F_i' also satisfies the same system of axioms. In other words, *if two fields with certain predicates are isomorphic and if one of them together with its predicates satisfies some system of axioms, then the other field also satisfies the same system of axioms.*

The converse question arises: if two fields with certain predicates satisfy the same system of axioms, will they be isomorphic? It is easily verified that the answer to this question is in the negative.

The system of order axioms which we considered above: (1) $\overline{x < x}$, (2) $x < y \to (y < z \to x < z)$ is an example of a system of axioms for which there exist interpretations which are not mutually isomorphic. In fact, this system of axioms with the predicate $x < y$ is satisfied by the field M consisting of the two elements a and b if we assume that $a < b$ is true but that $a < a$, $b < b$ and $b < a$ are false. Moreover, this system of axioms is satisfied by the field M' consisting of the three elements a, b, c if the predicate $x < y$ is defined for this field as this was done in §4. But the field M cannot be isomorphic to the field M' because these fields consist of a different number of elements.

Every system of axioms for which all interpretations are isomorphic is called a complete system.

THEOREM. *Suppose the system of axioms*

$$A_1, A_2, \ldots, A_k, A_{k+1}$$

has an interpretation and that the axioms

$$A_1, A_2, \ldots, A_k \tag{1}$$

constitute a complete system of axioms. If, in this connection, the axiom

A_{k+1} does not contain individual predicates which are distinct from those occurring in (1), *then the axiom A_{k+1} is dependent on the axioms in* (1).

Suppose the individual predicates $F_1, F_2, \ldots, F_p$ occur in the system of axioms (1). By assumption, the axiom A_{k+1} does not contain other individual predicates.

We have already shown that if the system of axioms is satisfied by some field with individual predicates, then it is also satisfied by every other field with individual predicates which is isomorphic to the first field. It follows from this that if the system of axioms is not satisfied by some field and predicates, then also it is not satisfied by any other field with predicates which is isomorphic to the first field.

By the condition of the theorem, there exists some interpretation of the system of axioms

$$A_1, A_2, \ldots, A_k, A_{k+1}. \tag{2}$$

This interpretation represents a field M' with predicates $F_1', \ldots, F_p'$ which are used to replace the predicates $F_1, \ldots, F_p$ in the axioms. This field, manifestly, also satisfies system (1). By virtue of the completeness of system (1), all its interpretations are isomorphic. And, since this system contains the same individual predicates as system (2), every interpretation of system (1) is isomorphic to the given interpretation of system (2). It follows from the properties of an isomorphism then that an arbitrary interpretation of system (1) also satisfies system (2). Thus, every interpretation of system (1) is also an interpretation of system (2). This means that the axiom A_{k+1} depends on the system of axioms (1), which is what we were required to prove.

Of course, for every consistent system of axioms, there exist axioms which are independent of it. One can take, for example, the formula $(x)F^*(x)$, where $F^*(x)$ is an individual predicate, which is not contained in the given system of axioms, and adjoin it to the system in the role of a new axiom. If our initial system is interpreted on the field M with the predicates

$$F_1, \ldots, F_p,$$

then, defining $F^*(x)$ on this field in such a way that the formula $(x)F^*(x)$ is false, we can conclude that the field with the predicates

$$F_1, \ldots, F_p, F^*$$

satisfies the initial system of axioms but that it does not satisfy the system obtained from it by the adjunction of the axiom $(x)F^*(x)$.

Furthermore, it is perfectly clear that an axiom which is not a tautological proposition on properties and relations of objects is independent of those axioms in which nothing is said of these properties and relations.

We have already had occasion to use the expression "completeness of a system of axioms" in another sense. In the sequel, we shall also use this expression in two senses. In order to distinguish these concepts, when we speak of them in one context, we shall call completeness, defined in this

section, *completeness to within isomorphism* or *formal completeness*, having in mind that we are here dealing with an isomorphism which retains those individual predicates which are described by the system of axioms under consideration.

§8. Axioms for the set of natural numbers

We shall abbreviate the predicate

$$x < y \vee x = y$$

in the form

$$x \leqslant y.$$

A predicate having the form

$$x < y \,\&\, (u)[u \leqslant x \vee y \leqslant u]$$

will be denoted by $\sigma(x, y)$. The meaning of the predicate $\sigma(x, y)$ can be expressed in words as follows: "y is the immediate successor of x" so that the truth of $\sigma(x, y)$ is equivalent to the assertion that y follows x and there is no object between them.

The symbol 0 denotes an individual object of the field.

AXIOMS

I

1. $x = x$,
2. $x = y \rightarrow (A(x) \rightarrow A(y))$.

II

1. $\overline{x < x}$,
2. $x < y \rightarrow (y < z \rightarrow x < z)$,
3. $(x)(\exists y)[\sigma(x, y) \,\&\, (u)(\sigma(x, u) \rightarrow u = y)]$.

III

$$A(0) \,\&\, (x)(y)[A(x) \,\&\, \sigma(x, y) \rightarrow A(y)] \rightarrow A(z).$$

As is clear from these axioms, the field characterized by them contains the individual object 0.

Axioms I characterize the equality of predicates.

The first two axioms of the second group define the predecessor predicate $<$. Axiom II.3 expresses the fact that for every object of the field there exists a unique object which is its immediate successor.

Axiom III is the axiom of complete induction. It consists in the assertion: "*if a proposition is true for the object* 0 *and if the fact that it is true for* x *implies that it is true for the immediately following object of the field then this proposition is true for arbitrary* z".

We note that the predicate $x = y$ in all interpretations cannot be anything

else than the identity of the objects x and y, inasmuch as only this predicate can satisfy the axioms I-III. It is easily verified that the sequence of natural numbers

$$0, 1, 2, \ldots, n, \ldots$$

(here and in the sequel we shall consider 0 to be a natural number), in which the predicate $x < y$ means that "the natural number x is less than the natural number y" in the usual sense of the word, satisfies axioms I, II, III. This system of axioms is thus formally consistent.

We now introduce a concept from set theory. We consider two ordered sets M_1 and M_2 on each of which there is defined a predicate which represents an order relation ($x_1 < y_1$ in the first and $x_2 < y_2$ in the second). We shall say that two ordered sets are similar if it is possible to establish a one-to-one correspondence $x_1 — x_2$ which leaves the order relations between elements invariant, i.e. if whenever $x_1 < y_1$ is true then $x_2 < y_2$ is also true for the corresponding elements x_2, y_2. In this case, in virtue of the fact that the sets under consideration can be ordered, if $x_1 < y_1$ is false then for the corresponding elements x_2, y_2, $x_2 < y_2$ is also false. In fact, if $x_1 < y_1$ is false, then either $x_1 = y_1$, or $y_1 < x_1$ is true. In the first case, $x_2 = y_2$, because our correspondence is one-to-one, and, hence, $x_2 < y_2$ is false. In the second case $y_2 < x_2$ is true. But then $x_2 < y_2$ is false because otherwise we would arrive at a contradiction with the order axioms.

We can thus state the following proposition: ordered sets are similar if and only if they, together with their order predicates, are isomorphic.

The sequence of natural numbers is an ordered set and satisfies axioms I, II, III. In all these axioms there occur only two individual predicates—the equality predicate and the order predicate. The predicate of equality is preserved under every one-to-one correspondence. Since the predicate of order is also preserved under a similarity mapping, similarity to the sequence of natural numbers is an isomorphism. It follows from this that every similarity and, consequently, any set which is isomorphic to the sequence of natural numbers satisfies the system of axioms I, II, III.

We shall prove the converse: *Every set with an order predicate satisfying axioms I, II, III defined on it, is similar to the sequence of natural numbers.*

In fact, suppose the field M satisfies axioms I, II, III. We consider, on the other hand, the sequence of natural numbers:

$$0, 1, 2, \ldots, n, \ldots$$

in which the order relation is the relation "n is less than m" in the usual sense of the word. To the natural number 0 we assign the element of the field M which is denoted by the same symbol. We denote this element by x_0 say. We set unity into correspondence with the element which follows immediately after x_0, denoting it by x_1. If we assign the element x_n to the number n, then to the number $n + 1$ we set into correspondence the element of the field

M which follows immediately after x_n. In virtue of axiom II.3, such an element exists and there is, moreover, only one. We denote it by x_{n+1}. We have thus assigned to each natural number n the element x_n of the field M. We obviously have that

$$x_0 < x_1; \quad x_1 < x_2; \quad \ldots; \quad x_n < x_{n+1}; \quad \ldots.$$

It follows from these relations that if $n < m$, then $x_n < x_m$.

This implies that the set of elements $x_0, x_1, \ldots, x_n, \ldots$, which we denote by M', is ordered and is similar to the set of natural numbers.

We shall show that this set comprises the whole field M, i.e. M' coincides with M.

The truth of this assertion will be proved by applying the axiom of complete induction which the field M satisfies. We state the proposition: "the element z from the field M belongs to the set M'". This proposition is true if this element is x_0 (or 0). If this proposition is true for some element x, then it is true also for the immediately following element. In fact, suppose x is any one of the elements x_n; then x_{n+1} is the element which follows immediately after x_n. On the basis of axiom II.3, for every element x_n there exists a unique element which is its immediate successor. In this case, the element x_{n+1} which is the immediate successor of the element x_n also belongs to M'. On the basis of the axiom of complete induction, we can now conclude that every element of the field M is also an element of the field M', i.e. M and M' coincide.

We have thus proved that every field which satisfies axioms I, II, III is ordered and is similar to the set of natural numbers. We can formulate the result thus obtained in the following way: every interpretation of axioms I, II, III is isomorphic to the set of natural numbers. From this it follows that distinct interpretations of axioms I, II, III are isomorphic. This means that the system of axioms under consideration is complete. One must furthermore assert that axioms I, II, III do not describe all the properties of the set of natural numbers. For instance, they do not contain the arithmetic operations of addition and multiplication. Axioms I, II and III define only the order relations of the set of natural numbers, and the completeness of this system of axioms is restricted in some sense. As we agreed above, it can be called complete to within an isomorphism which leaves the order relations invariant.

It can be proved that each of the axioms of the system under consideration is independent of the remaining ones. Here, we shall restrict ourselves to the proof of the independence of the axiom of complete induction. To this end we must find an interpretation which satisfies the system of axioms I and II and not satisfying the system consisting of all these axioms.

Let the field M be the set of rational numbers having the form $\frac{i}{i+1}$ and $1 + \frac{i}{i+1}$, where i takes on all possible values rom the set of natural

numbers. Thus, the set under consideration consists of the following numbers:

$$0, \tfrac{1}{2}, \tfrac{2}{3}, \ldots, \frac{i}{i+1}, \ldots; \quad 1, 1\tfrac{1}{2}, 1\tfrac{2}{3}, \ldots, 1 + \frac{i}{i+1}, \ldots.$$

Suppose an individual object, denoted in the system by the symbol "0", is the rational number 0 for a given field, and that the predicate $x < y$ denotes "the rational number x is less than the rational number y" in the usual sense of the word.

It is easily verified that the interpretation under consideration satisfies the system of axioms I and II.

In fact, for the predicate which denotes the identity of objects, axioms I are true for every replacement of the variable predicate A. Axioms II.1 and II.2 are also true because the predicate "the number x is less than the number y" is satisfied by them.

We now consider axiom II.3. It is asserted in this axiom that for every object there exists a unique immediate successor. But for numbers of the form $\frac{i}{i+1}$, the number $\frac{i+1}{i+2}$ is an immediate successor because $\frac{i}{i+1} < \frac{i+1}{i+2}$ and in our field M there do not exist numbers which are comprised between $\frac{i}{i+1}$ and $\frac{i+1}{i+2}$. Moreover, $\frac{i+1}{i+2}$ is the unique successor of $\frac{i}{i+1}$ since every number which is greater than $\frac{i}{i+1}$ and does not coincide with $\frac{i+1}{i+2}$ is greater than $\frac{i+1}{i+2}$, and therefore it cannot be the immediate successor of $\frac{i}{i+1}$. It is established in precisely the same way that the number $1 + \frac{i}{i+1}$ has the number $1 + \frac{i+1}{i+2}$ as its unique immediate successor. Thus the field M with the predicate $x < y$ satisfies the system of axioms I and II.

We shall show that under certain replacements of the predicate A and the object variable z, axiom III is false in our interpretation. We choose, for the replacement of the predicate A the predicate $A^0(x)$, having the value T for numbers of the form $\frac{i}{i+1}$ and F for numbers of the form $1 + \frac{i}{i+1}$. In this case, $A^0(0)$ is true.

$$A^0(x) \mathbin{\&} \sigma(x, y) \to A^0(y) \tag{3}$$

is also true for every x and y in our field. In fact, if $A^0(x)$ or $\sigma(x, y)$ is false, then formula (3) is true because the antecedent is false. We assume that $A^0(x)$ and $\sigma(x, y)$ are true. Then x is a number of the form $\frac{i}{i+1}$ and y is the number following immediately after it. But then y is the number $\frac{i+1}{i+2}$

which is a number of the same form. Therefore $A^0\left(\frac{i+1}{i+2}\right)$ is also true and, consequently, $A^0(y)$ is true. In this case, formula (3) is again true. Thus, formula (3) is true for every x and every y in the field M. Consequently, the formula

$$(x)(y)(A^0(x) \mathbin{\&} \sigma(x, y) \to A^0(y))$$

is also true, and, hence, the formula

$$A^0(0) \mathbin{\&} (x)(y)(A^0(x) \mathbin{\&} \sigma(x, y) \to A^0(y))$$

is true for the field M. But if z is a number of the form $1 + \frac{i}{i+1}$, then $A^0(z)$ is false and therefore the formula

$$A^0(z) \mathbin{\&} (x)(y)(A^0(x) \mathbin{\&} \sigma(x, y) \to A^0(y)) \to A^0(z)$$

is false if z is a number of the form $1 + \frac{i}{i+1}$.

We see that axiom III is false for some replacement of the variable predicate A and the object variable z, and therefore the field M does not satisfy the system of axioms I, II and III.

According to the definition of the independence of axioms we conclude that axiom III is independent of the remaining axioms of the system under consideration.

§9. Normal formulae and normal forms

As we have already seen (see §2), for every formula there exists a reduced formula which is equivalent to it. From among the reduced formulae we isolate a certain class of formulae which we shall call *normal formulae.*

A reduced formula is called normal if it does not contain quantifiers or if, in its formation from elementary formulae, the quantifiers are applied after all the operations of propositional algebra.

In writing down a normal formula, quantifiers—if there are any—precede all the remaining logical symbols. For example, the reduced formula

$$(x_1)(x_2)(\exists x_3)(x_4)A(x_1, \ldots, x_4)$$

is normal if $A(x_1, \ldots, x_4)$ does not contain any quantifiers.

THEOREM 1. *For every formula there exists a normal formula which is equivalent to it.*

Proof. The proof of this theorem is based on certain equivalences which are easily established.

1. $(x)A(x) \vee H$ is equivalent to $(x)(A(x) \vee H)$.

In fact, suppose the formula $(x)A(x) \vee H$ is true for some field M and for certain replacements of the free variables as objects as well as predicates.

Then either $(x)A(x)$ is true for these replacements on M or H is true. In the first case, $A(x)$ is true for every x belonging to M. But then

$$A(x) \vee H \tag{4}$$

is also true for every x in M and, consequently, the formula

$$(x)[A(x) \vee H] \tag{5}$$

is true.

In the second case, if H is true, then formulae (4) and (5) are also true on M for the given replacements of the free variables.

Suppose the formula $(x)A(x) \vee H$ is false for the given replacements. Then $(x)A(x)$ is false and H is false. Consequently, there exists an element x_0 in the field M such that $A(x_0)$ is false. But for this element $A(x_0) \vee H$ is false and, consequently, formula (5) is false.

2. The formula $(x)A(x) \,\&\, H$ is equivalent to $(x)[A(x) \,\&\, H]$.
3. The formula $(\exists x)A(x) \,\&\, H$ is equivalent to $(\exists x)[A(x) \,\&\, H]$.
4. The formula $(\exists x)A(x) \vee H$ is equivalent to $(\exists x)[A(x) \vee H]$.

The equivalences 2, 3 and 4 are proved like 1. Manifestly, the corresponding assertions for the case when the second summand (factor) is bound by the quantifier are also true.

The case when the formula contains constants is covered by the cases already considered inasmuch as constants a, b, c occurring in formulae A and H can be considered as replacements of the free variables y, z, t by the elements a, b, c.

We shall prove Theorem 1 by induction, following the law for the construction of formulae in predicate logic. For elementary formulae, represented either by the letters A, B, $\ldots$, or elementary predicates $A(x)$, $B(x, y)$, $\ldots$, our assertion is true because these formulae are themselves normal. Suppose for the formulae A_1 and A_2 there are normal formulae, $A_1^\bullet$ and $A_2^\bullet$ respectively. Let, for instance, $A_1^\bullet$ have the form

$$(x_1)(x_2) \ldots (\exists x_i) \ldots (\exists x_n)B_1(x_1, \ldots, x_n),$$

and let $A_2^\bullet$ have the form

$$(\exists y_1)(y_2) \ldots (y_m)B_2(y_1, \ldots, y_m).$$

We shall prove that for the formula $A_1 \vee A_2$ there also exists a normal formula which is equivalent to it. The formula $A_1^\bullet \vee A_2^\bullet$ is equivalent to the formula $A_1 \vee A_2$ but it is not, in general, a normal formula. However, using the equivalences proved above we can transform it into a normal formula.

First, the formula $A_1^\bullet \vee A_2^\bullet$ can be replaced by the formula

$$(x_1)[(x_2) \ldots (\exists x_n)B_1(x_1, \ldots, x_n) \vee (\exists y_1)(y_2) \ldots (y_m)B_2(y_1, \ldots, y_m)],$$

by placing $A_2^\bullet$ under the scope of the first quantifier of the formula $A_1^\bullet$. By virtue of 1 (above), this formula is equivalent to the formula $A_1^\bullet \vee A_2^\bullet$. We

can further carry out equivalent transformations within the scope of the quantifier (x_1) because as a result of binding two equivalent formulae by the quantifier with respect to the same variable x_1 we obtain two equivalent formulae also. For the same reason, formula $A_2^\bullet$ can be brought under the scope of the second quantifier (x_2), and so on. Let us assume that the formula $A_2^\bullet$ has already been brought under the scopes of all the quantifiers of the formula $A_1^\bullet$. Then we obtain the formula

$$(x_1)(x_2)\dots(\exists x_n)[B_1(x_1,\dots,x_n) \vee (\exists y_1)(y_2)\dots(y_m)B_2(y_1,\dots,y_m)].$$

In the same way, we can bring the summand $B_1(x_1,\dots,x_n)$ successively under all the quantifiers of the formula $A_2^\bullet$. This done, we obtain the normal formula

$$(x_1)(x_2)\dots(\exists x_n)(\exists y_1)(y_2)\dots(y_m)[B_1(x_1,\dots,x_n) \vee B_2(y_1,\dots,y_m)],$$

which is equivalent to the formula $A_1 \vee A_2$.

In an analogous manner, with the aid of the equivalences 3 and 4, we can construct a normal formula equivalent to the formula $A_1 \,\&\, A_2$ if the normal formulae $A_1^\bullet$ and $A_2^\bullet$, equivalent to A_1 and A_2 respectively, are known.

Let A^* be a normal formula equivalent to the formula A and suppose A^* has, for instance, the form

$$(x_1)(\exists x_2)(x_3)\dots(x_n)H(x_1,\dots,x_n).$$

The formula $\bar{A}^*$ is equivalent to the formula $\bar{A}$. But the formula $\bar{A}^*$ in turn is equivalent to the formula

$$(\exists x_1)(x_2)(\exists x_3)\dots(\exists x_n)\bar{H}(x_1,\dots,x_n),$$

which is a normal formula. Thus, knowing a normal formula which is equivalent to A, we can construct a normal formula which is equivalent to $\bar{A}$.

The formula $(x)A^*(x)$, obviously, is equivalent to the formula $(x)A(x)$, and the formula $(\exists x)A^*(x)$ is equivalent to the formula $(\exists x)A(x)$. But the formulae $(x)A^*(x)$ and $(\exists x)A^*(x)$ are normal formulae.

Thus, there exist equivalent normal formulae for elementary formulae; if the formula A is obtained with the aid of the operations &, $\vee$, — and quantifications from formulae for which there exist equivalent normal formulae, then there exists an equivalent normal formula for A also.

But since every formula of predicate algebra is obtained from elementary formulae with the aid of the indicated operations, for every formula of predicate algebra there exists a normal formula equivalent to it, which is what we required to prove.

A normal formula, equivalent to the formula A, will be called a *normal form of the formula A.*

§10. The decision problem

We shall pose this problem for formulae of the predicate calculus, *devoid of symbols for constant objects* and *symbols for individual predicates.* In the

subsequent discussion of this chapter, we shall assume that the formulae considered are of this sort (if it is not stated specifically otherwise).

Every such formula represents a definite assertion, true or false, when it is referred to a definite field M and the predicates occurring in it are replaced by individual predicates defined on M.

If such a formula is true for some field M and certain predicates, defined on M, we shall call it satisfiable.

If a formula is true for a given field M and for all predicates defined on M, we shall call it identically true for the field M.

If the formula is true for every field M and for all predicates, we shall call it *identically true* or *simply true.*

A formula is called false or unsatisfiable if it is not true for any field nor for any replacements of the predicates. It is easily shown that if the formula A is identically true, then the formula $\bar{A}$ is false, and conversely.

The formulation of the decision problem for predicate logic is analogous to the formulation of this problem for propositional algebra. It is posed in the following manner: Give an effective procedure for determining whether or not a given formula is satisfiable.

Knowing how to solve the problem of satisfiability, we are able by the same token to solve the problem of the truth of an arbitrary formula also. In fact, if the formula A is true, then the formula $\bar{A}$ is unsatisfiable, and conversely. Therefore, by proving the satisfiability or non-satisfiability of $\bar{A}$, we verify by the same token the truth of A. The decision problem for predicate logic is a strengthening of the decision problem for propositional calculus because all formulae in propositional calculus occur among the formulae of predicate logic. However, whereas the solution of the decision problem for propositional calculus offers no difficulty, the decision problem for predicate logic turns out to be connected with serious difficulties.

Modern investigations have shed light on the nature of these difficulties. At the present time it is known that the solution of this problem is not at all possible in the indicated sense. In other words, no constructive rule can exist which could enable one to determine for an arbitrary formula of the predicate logic whether it is identically true or not. It is sufficient to say that it is possible to point out a formula, the problem of the satisfiability of which is equivalent to the Fermat problem. For certain particular types of formulae, however, the decision problem can be solved. We shall consider the most important type of formulae for which the solution of the decision problem can be realized.

§11. The logic of predicates in one variable

We shall consider formulae in predicate logic containing predicates which depend on one variable only. The logic in which such expressions are used corresponds to the logic described by Aristotle and appeared as a traditional

element in the system of humanitarian education. The known forms of the propositions of this logic and the forms of inference—the so-called "syllogism modes"—are expressed completely in the symbolism of the logic of predicates of one variable.

THEOREM. *If a formula of predicate logic, containing only predicates of one variable, is satisfiable on some field M, then it is satisfiable on a field M' which contains no more than 2^n elements, where n is the number of predicates occurring in the formula under consideration.*

Proof. Suppose the formula $A(A_1, \ldots, A_n)$, containing only the predicate symbols $A_1, \ldots, A_n$, each of which depends on one variable, is satisfiable on some field M. We can assume that this formula is given in the normal form, and that all the object variables in it are bound. In fact, whatever the formula A is, we can, by carrying out the transformations indicated in §9 on it, transform it into a form in which all the quantifiers precede the remaining symbols of the formula, in which connection the supply of its predicates and object variables remains invariant. If A contains free object variables, then we can bind them with universal quantifiers.

Thus, we shall assume that A is a normal formula. Then we can write it in the following form:

$$(\sigma x_1)(\sigma x_2) \ldots (\sigma x_p)B(A_1, \ldots, A_n, x_1, \ldots, x_p),$$

where each of the symbols (σx_i) denotes either the quantifier (x_i) or $(\exists x_i)$, and the formula

$$B(A_1, \ldots, A_n, x_1, \ldots, x_p)$$

does not contain quantifiers.

In the formula $B(A_1, \ldots, A_n, x_1, \ldots, x_p)$ all the variables $x_1, \ldots, x_p$ occur in the predicates $A_1, \ldots, A_n$, and it can be written in the form

$$B(A_1(x_{i_1}), \ldots, A_n(x_{i_n})),$$

where $i_1, \ldots, i_n$ are the numbers from 1 to p. It will be more convenient, however, to make use of the expression

$$B(A_1, \ldots, A_n, x_1, \ldots, x_p),$$

if we keep in mind that B is a logical function of the predicates A_k and that each of the predicates A_k depends on some one of the variables x_{i_k}.

We shall show that if for some field M there exist individual predicates

$$A_1^0, \ldots, A_n^0$$

for which the formula $A(A_1^0, \ldots, A_n^0)$ is true, then this formula is also true on some subset of this field, which contains no more than 2^n elements. This will prove the theorem. We can assume that the field M contains more than 2^n elements for otherwise our assertion is trivial. We partition the elements of the set M into classes in the following way. For every sequence containing

n symbols T and F in arbitrary order, e.g. $(T, F, F, \ldots, T)$, there exists a subset (which may be empty) of the set M, containing those, and only those, elements x for which the sequence of predicate values $A_1^0(x), A_2^0(x), \ldots, A_n^0(x)$ coincides with the given sequence of symbols T and F.

We denote by

$$\delta_1, \delta_2, \ldots, \delta_n$$

a sequence of symbols T and F, where δ_i is T or F, and the class of elements x corresponding to this sequence will be denoted by

$$a_{\delta_1, \delta_2, \ldots, \delta_n}.$$

Certain of these classes can be empty inasmuch as it can occur that for some sequence $\delta_1, \ldots, \delta_n$ there does not exist an element for which the predicates $A_1^0, \ldots, A_n^0$ take on the corresponding values $\delta_1, \ldots, \delta_n$. Furthermore, each element of the set M belongs to one of the classes a, and distinct classes do not have elements in common. The number of all classes (empty and non-empty) is equal to the number of sequences $\delta_1, \ldots, \delta_n$, i.e. 2^n. Consequently, the number q of non-empty classes a does not exceed 2^n. We choose one element from each non-empty class and denote these elements by $a_1, a_2, \ldots, a_q$.

The set of all these elements will be denoted by M'. We shall prove that if the formula $A(A_1^0, \ldots, A_n^0)$ is true on the field M, then it is also true on the field M'. (Since M' is a part of the field M, the predicates A_i^0 are defined on M'.) To each element x of the field M we set into correspondence the element from M' belonging to the same class as x. In M' there exists one and only one such element. We denote the element from M' which is set into correspondence with x by $\phi(x)$. We can say that a function defined on the set M and taking on values in the set M' has been constructed.

It is easily seen that the following equivalence holds:

$$A_i^0(x) \sim A_i^0(\phi(x)).$$

In fact, $\phi(x)$ belongs to the same class a as does x. But, by definition, for elements of the same class, the predicate A_i^0 takes on the same value. It follows from this that if in the formula $A(A_1^0, \ldots, A_n^0)$ for every object variable t we replace each expression $A_i^0(t)$ by $A_i^0(\phi(t))$, then the formula $A(A_1^0, \ldots, A_n^0)$ goes over into the formula $A'(A_1^0, \ldots, A_n^0)$, which is equivalent to the first formula. The way formula A' is written differs from A only by the fact that all the object variables $x, y, z, \ldots, u$ of the formula A are replaced by $\phi(x), \phi(y), \ldots, \phi(u)$ respectively. This follows from the fact that, by assumption, the formula $A(A_1^0, \ldots, A_n^0)$ contains only the predicates A_i^0 and therefore every object variable occurs only under the sign of one of these predicates.

Suppose $R(x, y, \ldots, u)$ is a predicate defined on the field M. We introduce the notation

$$\widehat{(x)}R(x, y, \ldots, u).$$

By this expression we shall understand the predicate depending on $y, z, \ldots, u$ (or a proposition if $y, z, \ldots, u$ are absent) and taking on the value T when $R(x, y, z, \ldots, u)$ has the value T for the given $y, z, \ldots, u$ and for all x belonging to the field M', and taking on the value F otherwise. We also introduce the expression

$$\widehat{(\exists x)}R(x, y, \ldots, u),$$

which is a predicate of $y, \ldots, u$ and takes on the value T when $R(x, y, \ldots, u)$ has the value T for $y, \ldots, u$ and for at least one value x in the field M', and the value F otherwise. The signs $\widehat{(x)}$ and $\widehat{(\exists x)}$ will be called *restricted quantifiers*. If we bind all the variables of the predicate $R(x, y, \ldots, u)$ by restricted quantifiers, for instance,

$$\widehat{(x)}\widehat{(\exists y)} \ldots \widehat{(u)}R(x, y, \ldots, u),$$

we obtain a formula, referred to the field M'. We shall show that the expression

$$(x)R(\phi(x), y, \ldots, u)$$

is equivalent to the expression

$$\widehat{(x)}R(x, y, \ldots, u). \tag{6}$$

Suppose $(x)R(\phi(x), y, \ldots, u)$ has the value T. In this case,

$$R(\phi(x), y, \ldots, u)$$

has the value T for given $y, \ldots, u$ and for every x. But since the function $\phi(x)$ ranges over the entire field M' when x ranges over the field M, then $R(x, y, \ldots, u)$ has the value T for given $y, \ldots, u$ and for all x in M'. By virtue of the definition $\widehat{(x)}R(x, y, \ldots, u)$ also takes on the value T. Conversely, if $\widehat{(x)}R(x, y, \ldots, u)$ takes on the value T, then $R(x, y, \ldots, u)$ has the value T for given $y, \ldots, u$ and for every x in M'. In this case, the expression $R(\phi(x), y, \ldots, u)$ has the value T for given $y, \ldots, u$ and for every x in M, inasmuch as $\phi(x)$ belongs to M' for arbitrary x.

In an analogous manner, one can show that the expressions

$$(\exists x)R(\phi(x), y, \ldots, u) \text{ and } \widehat{(\exists x)}R(x, y, \ldots, u) \tag{7}$$

are also equivalent.

We consider next the formula $A(A_1^0, \ldots, A_n^0)$, which can be written in the form

$$(\sigma x_1)(\sigma x_2) \ldots (\sigma x_p)B(A_1^0, \ldots, A_n^0, x_1, \ldots, x_p).$$

Here,

$$B(A_1^0, \ldots, A_n^0, x_1, \ldots, x_p)$$

is a predicate defined on the field M and depending on p variables $x_1, \ldots, x_p$. Each of these variables occurs in the formula B only through the predicates $A_1^0, \ldots, A_n^0$. On the other hand, we saw that the predicates $A_i^0(x)$ and

$A_i^0(\phi(x))$ are equivalent. Therefore, if we replace x_i by $\phi(x_i)$ in the formula $\boldsymbol{B}(A_1^0, \ldots, A_n^0, x_1, \ldots, x_p)$, we obtain the equivalent expression:

$$\boldsymbol{B}(A_1^0, \ldots, A_n^0, x_1, \ldots, x_p) \sim \boldsymbol{B}(A_1^0, \ldots, A_n^0, \phi(x_1), \ldots, \phi(x_p)).$$

From this it follows that

$$(\sigma x_p)\boldsymbol{B}(A_1^0, \ldots, A_n^0, x_1, \ldots, x_p) \sim (\sigma x_p)\boldsymbol{B}(A_1^0, \ldots, A_n^0, \phi(x_1), \ldots, \phi(x_p)).$$

Further, one can conclude that

$$(\sigma x_p)\boldsymbol{B}(A_1^0, \ldots, A_n^0, \phi(x_1), \ldots, \phi(x_p)) \sim$$
$$\sim (\widehat{\sigma x_p})\boldsymbol{B}(A_1^0, \ldots, A_n^0, \phi(x_1), \ldots, \phi(x_{p-1}), x_p).$$

In fact, fixing the variables $x_1, \ldots, x_{p-1}$ in an arbitrary manner in the expression $\boldsymbol{B}(A_1^0, \ldots, A_n^0, \phi(x_1), \ldots, \phi(x_p))$, we obtain a predicate of one variable x_p. Applying the equivalence (6) or (7) to it, we obtain the equivalence required. It follows from the last two equivalences that

$$(\sigma x_p)\boldsymbol{B}(A_1^0, \ldots, A_n^0, x_1, \ldots, x_p) \sim$$
$$\sim (\widehat{\sigma x_p})\boldsymbol{B}(A_1^0, \ldots, A_n^0, \phi(x_1), \ldots, \phi(x_{p-1}), x_p).$$

Using an analogous line of reasoning, we obtain

$$(\sigma x_{p-1})(\sigma x_p)\boldsymbol{B}(A_1^0, \ldots, A_n^0, x_1, \ldots, x_{p-1}, x_p) \sim$$
$$\sim (\widehat{\sigma x_{p-1}})(\widehat{\sigma x_p})\boldsymbol{B}(A_1^0, \ldots, A_n^0, \phi(x_1), \ldots, \phi(x_{p-2}), x_{p-1}, x_p)$$

and, finally, we reach the following:

$$(\sigma x_1) \ldots (\sigma x_p)\boldsymbol{B}(A_1^0, \ldots, A_n^0, x_1, \ldots, x_p) \sim$$
$$\sim (\widehat{\sigma x_1}) \ldots (\widehat{\sigma x_p})\boldsymbol{B}(A_1^0, \ldots, A_n^0, x_1, \ldots, x_p).$$

The right member of the last equivalence, according to the meaning of the symbol $(\widehat{\sigma x})$, is nothing other than the formula

$$(\sigma x_1) \ldots (\sigma x_p)\boldsymbol{B}(A_1^0, \ldots, A_n^0, x_1, \ldots, x_p),$$

referred to the field $\boldsymbol{M}'$.

We have thus proved that the formula $A(A_1^0, \ldots, A_n^0)$ retains its value if it is referred to the field $\boldsymbol{M}'$, and the theorem is thus proved.

COROLLARY. *If a formula A, containing only predicates which depend on one variable, is identically true for every field having not more than 2^n elements, where n is the number of predicates in A, then the formula A is identically true* (i.e. it is true for an arbitrary field). In fact, let us assume that A is not an identically true formula. In this case, its negation $\bar{A}$ is satisfiable on some field. Since $\bar{A}$ also satisfies the conditions of the theorem, a field can be found containing no more than 2^n elements on which the formula $\bar{A}$ is satisfiable. Consequently, A cannot be true on this field, which contradicts our assumption. Thus, the assumption that A is not identically true leads to a contradiction, which is what we were required to prove.

The theorem just proved enables one to solve the decision problem for formulae containing only predicates depending on one variable. It is clear from the corollary that in order to establish whether or not the formula A is identically true, it suffices to verify whether or not it is identically true for every field containing no more than 2^n elements.

We note that it suffices to verify whether or not a given formula A is identically true on a field consisting of exactly 2^n elements. This follows from the fact that for formulae of the type under consideration the following holds: If a formula A is identically true on some field, then it is identically true on each of its parts.

We consider an arbitrary field containing exactly 2^n elements: $x_1, x_2, \ldots, x_{2^n}$. It is easy to see that every formula having the form $(x)B(x)$, referred to the given field, is equivalent to the formula

$$B(x_1) \,\&\, B(x_2) \,\&\, \ldots \,\&\, B(x_{2^n}).$$

And a formula having the form $(\exists x)B(x)$ is equivalent to the formula

$$B(x_1) \vee B(x_2) \vee \ldots \vee B(x_{2^n}).$$

In such a case, an arbitrary formula A referred to the field

$$\{x_1, \ldots, x_{2^n}\}$$

is equivalent to the formula A' in which all the quantifiers are replaced by operations of logical product and logical sum. Since only the predicates $A_1, \ldots, A_n$, depending on one variable, occur in A, then A' is a formula formed by only the operations of propositional algebra on the expressions $A_i(x_j)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant 2^n$. Since the predicates $A_i(x)$ are perfectly arbitrary, the expressions $A_i(x_j)$ are perfectly arbitrary propositions. The formula A' can then be considered as a formula of propositional algebra in which the $A_i(x_j)$ are elementary variable propositions. Then the problem of whether or not A is identically true on the field $x_1, x_2, \ldots, x_{2^n}$ turns out to be equivalent to the problem whether or not A' is identically true, as a formula of propositional algebra with the propositional variables $A_i(x_j)$.

We note that a formula A' of propositional algebra does not depend essentially on the nature of the elements of the field $\{x_1, \ldots, x_p\}$ and depends only on their number, for if we take another field $\{x_1', \ldots, x_p'\}$ then in A' there occurs only a change in the notation for the propositional variables $A_i(x_j)$ to $A_i(x_j')$. By virtue of this, we can say that if A' is identically true, as a formula in propositional algebra, then the formula A is identically true on an arbitrary field of p elements, and conversely. On the other hand, in Chapter I we arrived at a constructive procedure for determining whether or not an arbitrary formula of propositional algebra is identically true. Applying this criterion, we can establish whether an arbitrary formula A containing only predicates of one variable is identically true on an arbitrary field con-

taining $p = 2^n$ elements. In this case, by virtue of the assumption we stated above we can also solve the problem of whether a formula A is identically true or not.

§12. Finite and infinite fields

In order to clarify the question of which fields can be characterized by axioms containing only predicates of one variable, we must strengthen the theorem of the preceding section somewhat by proving the following theorem.

THEOREM. *Let A be a formula which contains no free object variables, contains only individual objects $c_1, \ldots, c_q$, individual predicates $A_1, \ldots, A_n$ and predicate variables $B_1, \ldots, B_m$, where the predicates A_i as well as the predicates B_j depend only on one variable; and suppose that on some field M with the individual predicates $A_1^0, \ldots, A_n^0$ defined on it, the formula A is true for arbitrary replacements of the symbols $B_1, \ldots, B_m$ by predicates. Then there exists a field M' containing no more than 2^n elements, with the individual predicates $A_1', \ldots, A_n'$ defined on it, on which the formula A is also true.*

The expression "the formula A is true on the field M with the predicates A_i" is used here in the same sense as in §4. Except that in the present case the system of axioms consists of the one axiom A. The situation is not essentially different because we can always replace an arbitrary system of axioms by a single axiom combining the axioms of the given system by the sign &. Suppose we are given the system of axioms

$$A_1, A_2, \ldots, A_n.$$

Then the system consisting of the one axiom

$$A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n$$

is satisfied by the same fields and predicates as the system $A_1, A_2, \ldots, A_n$, which fact is directly clear from the meaning of the operation &.

We now proceed to the proof of the theorem. We consider the classes α defined as was done in the preceding section in connection with the predicates $A_1^0, \ldots, A_n^0$. Suppose $a_1, \ldots, a_p$ are elements chosen from each non-empty class α. The set of all these elements will be denoted by M'. On the field M we consider predicates B' of one variable which for elements belonging to the same class α have the same value. Let $\phi(x)$, as in the preceding theorem, represent an element a_i belonging to the class α in which x occurs. We already know that

$$A_i^0(x) \sim A_i^0(\phi(x)).$$

By definition, the predicates $B'(x)$ also possess this property, i.e.

$$B'(x) \sim B'(\phi(x)),$$

because x and $\phi(x)$ always belong to the same class α. Then, following the

same line of reasoning as in the proof of the preceding theorem, we can show that if the formula

$$A(c_1, \ldots, c_q, A_1, \ldots, A_n, B_1, \ldots, B_m)$$

for certain definite replacements of the objects c and the predicates A and B by objects and predicates from M is true, then the formula

$$A(\phi(c_1), \ldots, \phi(c_q), A_1, \ldots, A_n, B'_1, \ldots, B'_m)$$

is also true.

By assumption, the formula A satisfies the field M with the predicates $A_1^0, \ldots, A_n^0$. Therefore there exist objects $c_1^0, \ldots, c_q^0$ such that the formula

$$A(c_1^0, \ldots, c_q^0, A_1^0, \ldots, A_n^0, B_1, \ldots, B_m)$$

is true for all replacements of the predicates B by predicates defined on M. In this case, the formula

$$A(c_1^0, \ldots, c_q^0, A_1^0, \ldots, A_n^0, B'_1, \ldots, B'_m)$$

is true for all possible predicates B'. It follows from what we have just said that the formula

$$A(\phi(c_1^0), \ldots, \phi(c_q^0), A_1^0, \ldots, A_n^0, B'_1, \ldots, B'_m)$$

is true on the field M' for all possible predicates B'. But the set of predicates B' on the field M' coincides with the set of all predicates of this field inasmuch as the values of the predicates B' for elements of distinct classes α are not connected by any condition, and in M' from every class α there is present only one element. Therefore the formula

$$A(\phi(c_1^0), \ldots, \phi(c_q^0), A_1^0, \ldots, A_n^0, B_1, \ldots, B_m)$$

is true for all possible B from the field M': consequently, the field M' with the predicates $A_1^0, \ldots, A_n^0$ satisfies this formula. This completes the proof of the theorem.

We shall now consider the problem of which sets can be characterized by axioms containing predicates of only one variable. Suppose

$$A_1, A_2, \ldots, A_n$$

is a system of axioms containing predicates of only one variable. As we indicated in §4, it is always possible to assume that the axioms A_i do not contain free variables. Moreover, this system of axioms can be replaced by one axiom:

$$A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n,$$

which we shall denote by the letter A without an index. It follows from the theorem just proved that if the formula A is interpreted by any field then it is also interpretable by a finite field. The same is true also for the system of axioms $A_1, A_2, \ldots, A_n$.

It follows from all this that no compatible system of axioms containing

predicates of only one variable exists such that the infiniteness of the field characterized by it could be deduced from it.

In other words, by means of axioms with predicates depending on only one variable it is impossible to distinguish an infinite set from a finite set. This means that it is impossible to distinguish an infinite set from a finite one if we make statements about the properties of its elements only and not about the relations between them. Consequently, in order to characterize such sets as, for example, the set of natural numbers, the set of all real numbers, without which mathematics is unthinkable, it is necessary to utilize predicates depending on more than one variable.

Axioms, in which predicates depending on an arbitrary number of variables are admitted, can characterize infinite sets. The system of axioms for the set of natural numbers, given in §8, can serve as an example. A field satisfying these axioms is similar to the set of natural numbers and cannot therefore be finite.

We can introduce a simple formula which is not satisfiable on a finite field but which is satisfiable on an infinite field:

$$(x)(y)(z)(\exists u)[\overline{F}(x, x) \;\&\; (F(x, y) \to (F(y, z) \to F(x, z))) \;\&\; F(x, u)].$$

Let us assume that this formula is satisfied on some field M. In this case there must exist a predicate $F^0(x, y)$ for which this formula is true on the field M. It is not difficult to verify that in this case the predicate $F^0(x,y)$ represents a predicate which establishes an order relation between the elements of the field M. It is evident from the formula that the predicate $F^0(x, y)$ does in fact satisfy the order relations:

1. $\overline{F}^0(x, x)$,
2. $F^0(x, y) \to (F^0(y, z) \to F^0(x, z))$.

We agree to express $F^0(x, y)$ by the words "x precedes y". However, as is obvious from the formula, for every x there must exist a u such that $F(x, u)$ is true, or "x precedes u". We take an arbitrary element of the field x_1; among the elements of the field we must be able to find an element x_2 such that "x_1 precedes x_2". In exactly the same way, an element x_3 can be found such that "x_2 precedes x_3" and so on. We obtain the sequence of elements: $x_1, x_2, \ldots, x_n$.

By virtue of axioms 1 and 2, every element of this sequence is different from every element with smaller index because "x_1 precedes x_n", "x_2 precedes x_n", ..., "x_{n-1} precedes x_n" will hold. But this signifies that any two elements of our sequence are distinct and the field M is infinite. We have thus proved that if the formula under consideration is satisfied on some field, then this field is infinite.

We shall now show that there exists a field on which the given formula is satisfied. Suppose M is the set of natural numbers and that $F(x, y)$ signifies

that "x is greater than or equal to y". Then $\overline{F}(x, y)$ means $x < y$. With such a replacement of the predicate $F(x, y)$, the formula takes on the form:

$$(x)(y)(z)(\exists u)[\overline{x < x} \mathbin{\&} ((x < y) \rightarrow (y < z \rightarrow x < z)) \mathbin{\&} x < u].$$

It is easy to see that this expression is in fact true for the set of natural numbers.

In order to consider certain problems which touch upon formulae satisfiable on infinite fields, it is necessary for us to introduce a concept from set theory which represents a generalization of the concept of "the number of elements in a finite set" and which enables one to distinguish infinite sets so that this distinction is not connected with either the nature of the elements or with the relations between the elements of the sets under consideration.

We shall call two sets M and M' equipollent if we can establish a one-to-one correspondence between their elements. By virtue of this definition, sets fall into classes of mutually equipollent sets, and the expression "the power of a given set" or "the number of elements in the given set" signifies the membership of the given set in one or another class.

Sets which are equipollent to the set of natural numbers are called *countable* sets. Infinite sets which are not equipollent to the set of natural numbers are called *non-countable* sets. It is proved in set theory that non-countable sets exist; thus, for example, the set of real numbers is a non-countable set.

Countable sets have the smallest power among all infinite sets. (The meaning of the expression "the power of the set M is less than the power of the set M'" is defined in set theory in the following way: the power of M is less than the power of M' if M and M' are not equipollent and M is equipollent to a subset of M'.)

For the satisfiability of formulae in predicate logic the following theorem which we shall prove in §14 holds.

LÖWENHEIM'S THEOREM. *If a formula is satisfiable on some infinite set then it is satisfiable on a countable set also.*

§13. Decision functions (Skolem's functions)

We consider a formula of the form

$$(\exists x_1) \ldots (\exists x_n)(y_1) \ldots (y_i) \ldots (\exists z_1) \ldots (u_1) \ldots$$
$$\ldots (\exists v_1) \ldots B(x_1, \ldots, y_1, \ldots, v_1, \ldots), \qquad (1)$$

where B is an individual predicate defined on some field M, and we shall assume that all the variables in formula (1) are bound. The distribution and number of quantifiers occurring before B is perfectly arbitrary, and the fact that in expression (1) we placed an existential quantifier first in writing the quantifiers is of no significance. However, without loss of generality, we can assume that in expression (1) the existential quantifier always occurs, and even, *a fortiori*, we can assume that the last quantifier written down is the

existential quantifier. To this end, it is only necessary to note that if $I(t)$, where t does not occur free in $\boldsymbol{B}$, represents an identically true predicate on the field $\boldsymbol{M}$, then

$$\boldsymbol{B}(x_1, \ldots, v_1, \ldots) \sim (\exists t)(\boldsymbol{B}(x_1, \ldots, v_1, \ldots) \,\&\, I(t))$$

holds for all values of the variables occurring in $\boldsymbol{B}$. In this case, if in formula (1) the last quantifier is the universal quantifier, we can replace the expression $\boldsymbol{B}$ by the equivalent expression $(\exists t)(\boldsymbol{B} \,\&\, I(t))$. Then we obtain a formula in which the last quantifier is the existential quantifier. In the sequel, we shall then assume that the last quantifier is the existential quantifier.

We shall partition the quantifiers in formula (1) into groups so that quantifiers of the same type which immediately follow one another belong to the same group (existential or universal). Thus, in expression (1) we shall have the following groups:

first group

$$(\exists x_1), \ldots, (\exists x_{n_1}),$$

second group

$$(y_1), \ldots, (y_{n_2}),$$

third group

$$(\exists z_1), \ldots, (\exists z_{n_3}),$$

and so on.

To each variable which is bound by an existential quantifier we set into correspondence some function which is defined on the field $\boldsymbol{M}$, taking on values from the field $\boldsymbol{M}$ and depending only on the variables which are bound by universal quantifiers which precede the given existential quantifier. But if the existential quantifier is not preceded by any universal quantifier, then we assign to it any individual object of the field. (One can say that in this case the function degenerates into a constant.) For formula (1), to the variables x_i will thus be set into correspondence certain individual objects x_i^0, to the variables z_i functions $\phi_i(y_1, \ldots, y_{n_2})$, and so on, to the variables v_i functions $\psi_i(y_1, \ldots, y_{n_2}, \ldots, u_1, \ldots, u_{n_{k-1}})$. If these objects and functions are such that, as the result of the replacement by them of the corresponding variables in the predicate

$$\boldsymbol{B}(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}, \ldots),$$

the predicate

$$\begin{aligned} \boldsymbol{B}(x_1^0, \ldots, x_{n_1}^0, y_1, \ldots, y_{n_2}, \phi_1(y_1, \ldots, y_{n_2}), \ldots, \phi_{n_3}(y_1, \ldots, y_{n_2}), \ldots \\ \ldots, u_1, \ldots, u_{n_{k-1}}, \psi_{n_k}(y_1, \ldots, u_1, \ldots, u_{n_{k-1}}), \ldots) \end{aligned} \tag{2}$$

obtained turns out to be true for all values of the variables occurring in it, then we will call the indicated objects and functions, viz.

$$x_1^0, \ldots, x_{n_1}^0, \phi_1(y_1, \ldots, y_{n_2}), \ldots, \psi_{n_k}(y_1, \ldots, u_{n_{k-1}})$$

decision functions (*or Skolem functions*) *for formula* (1). In this connection,

this expression will also refer to the objects x_i^0 so that it will not be necessary to introduce a special term for them.

LEMMA. *A necessary and sufficient condition that formula* (1) *be true on the field* M *is that there exist decision functions for this formula.*

First, it is evident that for (1) to be true it is necessary and sufficient that there exist objects $x_1^0, \ldots, x_{n_1}^0$ in the field M such that

$$(y_1) \ldots (y_{n_2})(\exists z_1) \ldots (\exists v_{n_k})B(x_1^0, \ldots, x_{n_1}^0, y_1, \ldots, v_{n_k})$$

is a true formula on M.

We shall prove the lemma by induction on the number of groups of quantifiers.

The validity of the lemma for the case when there is only one group of quantifiers in formula (1) follows from the remarks made at the beginning of this section. In fact, in this case the formula has the form

$$(\exists x_1) \ldots (\exists x_{n_1})B(x_1, \ldots, x_{n_1}),$$

because, according to our assumption, the last quantifier must be an existential quantifier. Obviously, in this case a necessary and sufficient condition for the formula to be true is that elements

$$x_1^0, x_2^0, \ldots, x_{n_1}^0$$

exist in the field M such that the formula

$$B(x_1^0, x_2^0, \ldots, x_{n_1}^0)$$

is true. In this case, a solution of the system of functions is the system of elements $x_1^0, \ldots, x_{n_1}^0$.

We shall assume that formula (1) has $p + 1$ groups of quantifiers. Two cases are possible:

(1) the first group consists of existential quantifiers;

(2) the first group consists of universal quantifiers.

In the first case, formula (1) has the form

$$(\exists x_1) \ldots (\exists x_{n_1})(y_1) \ldots (y_{n_2}) \ldots B(x_1, \ldots, x_{n_1}, y_1, \ldots). \tag{3}$$

A necessary and sufficient condition for this formula to be true is that there exist objects

$$x_1^0, x_2^0, \ldots, x_{n_1}^0$$

in the field M for which

$$(y_1) \ldots (y_{n_2}) \ldots B(x_1^0, \ldots, x_{n_1}^0, y_1, \ldots)$$

is true. But this last formula itself is a formula of form (1) and already contains only p groups of quantifiers. On the basis of the induction assumption, a necessary and sufficient condition for the truth of this formula is the

existence of a system of decision functions

$$\phi_1(y_1, \ldots, y_{n_2}), \ldots, \phi_{n_3}(y_1, \ldots, y_{n_2}), \ldots \\ \ldots, \psi_{n_k}(y_1, \ldots, y_{n_2}, \ldots, u_{n_{k-1}}), \tag{4}$$

i.e. of a system of function such that the expression

$$B(x_1^0, \ldots, x_{n_1}^0, y_1, \ldots, y_{n_2}, \phi_1(y_1, \ldots, y_{n_2}), \ldots, \psi_{n_k}(y_1, \ldots, u_{n_{k-1}})) \tag{5}$$

is true for all values of the variables on the field M. But then a necessary and sufficient condition for the truth of formula (3) is the existence of objects

$$x_1^0, x_2^0, \ldots, x_{n_1}^0$$

and functions

$$\phi_1(y_1, \ldots, y_{n_2}), \ldots, \psi_{n_k}(y_1, \ldots, u_{n_{k-1}})$$

for which expression (5) is true for all values of the variables. But this means that the objects $x_1, \ldots, x_{n_1}$ and functions (4) form a system of decision functions for formula (1).

We have thus proved that in the case under consideration a necessary and sufficient condition for the truth of formula (1) is that there exist a system of decision functions.

We now proceed to the second case—when the first group of quantifiers consists of universal quantifiers. In this case, formula (1) has the form

$$(x_1) \ldots (x_{n_1})(\exists y_1) \ldots (\exists y_{n_2}) \ldots B(x_1, \ldots, y_1, \ldots, v_{n_k}), \tag{6}$$

where the number of groups of quantifiers in this formula equals, by hypothesis, $p + 1$.

A necessary and sufficient condition for this formula to be true is that the formula

$$(\exists y_1) \ldots (\exists y_{n_2}) \ldots B(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}, \ldots, v_{n_k})$$

be true for all values of the variables $x_1, \ldots, x_{n_1}$ in M. We take an arbitrary group of values of the variables in the field M:

$$x_1^0, x_2^0, \ldots, x_{n_1}^0.$$

The formula

$$(\exists y_1) \ldots (\exists y_{n_2})(z_1) \ldots B(x_1^0, \ldots, x_{n_1}^0, y_1, \ldots, y_{n_2}, \ldots, v_{n_k}) \tag{7}$$

for values of the variables $x_1, \ldots, x_{n_1}$ fixed in this manner, does not have free object variables and represents an expression of the form (1). The number of distinct groups of quantifiers of the same type (i.e. existential or universal) in this formula equals p. According to the induction hypothesis, a necessary and sufficient condition for this formula to be true is that there exist a system of decision functions

$$y_1^0, \ldots, y_{n_2}^0, \chi_1(z_1, z_2, \ldots, z_{n_3}), \ldots, \psi_{n_k}(z_1, \ldots, u_{n_{k-1}}).$$

We shall assume that formula (6) is true. Then for every set of values of the variables $x_1^0, \ldots, x_{n_1}^0$ there must exist a solution system of formula (7).

We choose for each set of values of the variables $x_1^0, \ldots, x_{n_1}^0$ a completely well-defined solution system of formula (7) and denote it by

$$\begin{gathered} y_1^0(x_1^0, \ldots, x_{n_1}^0), \ldots, y_{n_2}^0(x_1^0, \ldots, x_{n_1}^0), \\ \chi_1(x_1^0, \ldots, x_{n_1}^0, z_1, \ldots, z_{n_3}), \ldots . \end{gathered} \tag{8}$$

The expression

$$B(x_1^0, \ldots, x_{n_1}^0, y_1^0(x_1^0, \ldots, x_{n_1}^0), \ldots, \psi_{n_k}(x_1^0, \ldots, x_{n_1}^0, z_1, \ldots))$$

is true for all values of the free variables occurring in it whatever the

$$x_1^0, x_2^0, \ldots, x_{n_1}^0$$

from the field M are. But we can consider $y_i^0(x_1^0, \ldots, x_{n_1}^0)$ as a function defined on the field M and taking on values from M and depending on the variables $x_1, \ldots, x_n$. We denote it by $\phi_i(x_1, \ldots, x_n)$. In exactly the same way we can consider $\chi_i(x_1^0, \ldots, x_{n_1}^0, z_1, \ldots, z_{n_3})$ as a function depending on the variables $x_1, \ldots, x_{n_1}, z_1, \ldots, z_{n_3}$, and write it in the form

$$\chi_i(x_1, \ldots, x_{n_1}, z_1, \ldots, z_{n_3}).$$

And so on for all functions of system (8).

In this case, the predicate

$$B(x_1, \ldots, x_{n_1}, \phi_1(x_1, \ldots, x_{n_1}), \ldots, \psi_{n_k}(x_1, \ldots, u_{n_{k-1}}))$$

is true for all values of the variables occurring in it. But then the system of functions

$$\phi_1(x_1, \ldots, x_{n_1}), \ldots, \phi_{n_2}(x_1, \ldots, x_{n_1}), \ldots, \psi_{n_k}(x_1, \ldots, u_{n_{k-1}}) \tag{9}$$

is a decision system for formula (6). (It is easy to see that the dependence of the functions occurring in this system on the variables is such as is required for a decision system.)

Conversely, let us assume that for formula (6) there exists a solution system of functions (9). In this case,

$$B(x_1, \ldots, x_{n_1}, \phi_1(x_1, \ldots, x_{n_1}), \ldots, \psi_{n_k}(x_1, \ldots, u_{n_{k-1}}))$$

is true for all values of the variables occurring. Then for every set of values of the variables $x_1^0, \ldots, x_{n_1}^0$,

$$B(x_1^0, \ldots, x_{n_1}^0, \phi_1(x_1^0, \ldots, x_{n_1}^0), \ldots, \psi_{n_k}(x_1^0, \ldots, u_{n_{k-1}}))$$

is true, and, consequently, the system of functions

$$\phi_1(x_1^0, \ldots, x_{n_1}^0), \ldots, \psi_{n_k}(x_1^0, \ldots, x_{n_1}^0, z_1, \ldots, u_{n_{k-1}}),$$

obtained from the given system of functions by fixing the variables $x_1, \ldots, x_{n_1}$, is a decision system for formula (7).

On the basis of what was stated above, we can conclude then that formula (7) will also be true for arbitrary values of the variables $x_1, \ldots, x_{n_1}$ and formula (6) will be true. Since the lemma is valid in the case when there is

only one group of quantifiers in formula (1), and, being true for p groups of quantifiers, it remains true for $p + 1$ groups, the lemma is completely proved.

§14. Löwenheim's theorem

All discussion in the preceding section is independent of whether or not individual objects occur in the formulae under consideration. Therefore, the assertions proved in that section can also be applied to satisfiability in the extended sense of the word, which we shall now define.

The concept of the satisfiability of a formula was defined in §10 only for formulae which do not contain symbols of individual objects and predicates. We shall extend the concept of satisfiability so that it is applicable to the case when the formula contains symbols of individual objects.

Suppose the formula A satisfies all the conditions stated earlier, except the requirement that symbols of individual objects be absent. We shall call a formula satisfiable on the field M if all the predicates occurring in A can be replaced by predicates defined on M and the symbols of individual objects by objects from the field M in such a way that the formula thus obtained is true.

LÖWENHEIM'S THEOREM. *If a formula which does not contain free object variables (but perhaps contains symbols of individual objects) is satisfiable for some field, then it is satisfiable on a finite or denumerable field.*

In the proof of this theorem we can limit ourselves to the consideration of normal formulae inasmuch as the normal form of an arbitrary formula and the formula itself are simultaneously satisfiable or non-satisfiable on every field. We consider an arbitrary normal formula all of whose object variables are bound but, generally speaking, containing symbols of individual objects. For example,

$$(x_1) \dots (x_{n_1})(\exists y_1) \dots (\exists y_{n_2}) \dots B(x_1, \dots, a_1, \dots, a_p, A_1, \dots, A_q), \quad (1)$$

where $a_1, \dots, a_p$ are symbols of individual objects, and $A_1, \dots, A_q$ are the elementary predicates which occur in the formula. The fact that the universal quantifier appears first is of no significance for our line of reasoning. We shall assume that this formula is satisfiable on some field M. In this case, objects $a_1^0, \dots, a_p^0$ belonging to the given field and predicates $A_1^0, \dots, A_q^0$ defined on this field can be found such that the formula

$$(x_1) \dots (x_{n_1}) \dots B(x_1, \dots, x_{n_1}, \dots, a_1^0, \dots, a_p^0, A_1^0, \dots, A_q^0) \quad (2)$$

is a true proposition.

Applying the conditions for truth, proved in the preceding section, we can say that for formula (2) there exists a system of decision functions

$$\phi_i(x_1, \dots, x_{n_1}), \psi_j(x_1, \dots, x_{n_1}, z_1, \dots, z_{n_3}), \dots .$$

In this case, the expression

$$B(x_1, \ldots, x_{n_1}, \phi_1(x_1, \ldots, x_{n_1}), \ldots, z_1, \ldots \\ \ldots, z_{n_3}, \psi_1(x_1, \ldots, x_{n_1}, z_1, \ldots, z_{n_3}), \ldots \\ \ldots, a_1^0, \ldots, a_p^0, A_1^0, \ldots, A_q^0)$$

is a predicate which takes on the value T for all values of the variables occurring in it.

For the sake of brevity, we shall denote this predicate by

$$T(x_1, \ldots, x_{n_1}, z_1, \ldots, z_{n_3}, \ldots),$$

where the variables $x_1, \ldots, x_{n_1}, z_1, \ldots$ are variables bound in formula (2) by universal quantifiers. We construct a certain set M' which is a subset of the field M. We define this set as the set-theoretic sum of a sequence of finite sets

$$M_1', M_2', \ldots, M_n', \ldots$$

The set M_1' consists of all individual elements $a_1^0, \ldots, a_p^0$, and if the formula (2) does not contain individual objects, then M_1' consists of one element chosen at random from the field M.

We shall assume that we have defined the set M_n'. We then define the set M_{n+1}'. Included in M_{n+1}' will be, first, all elements of the set M_n' and we then adjoin to it all those values of each of the decision functions which it takes on for all possible replacement of its object variables from the set M_n'. Obviously, if the set M_n' is finite, then the set M_{n+1}' is also finite. We thus define a sequence of finite sets

$$M_1', M_2', \ldots, M_n', \ldots$$

Their set-theoretic sum represents a denumerable or finite set. This assertion is a consequence of known theorems in set theory but it is also perfectly manifest without them.

We denote the set-theoretic sum

$$M_1' \cup M_2' \cup \ldots \cup M_n' \cup \ldots \tag{3}$$

by M'. The set M' possesses the following property: If the arguments of any of the decision functions

$$\ldots, \phi_i, \ldots, \psi_j, \ldots$$

take on values from M', then the value of the function itself is also an element in M'. Suppose $\theta(x_1, \ldots, x_{n_1}, z_1, \ldots, z_{n_3}, \ldots)$ is an arbitrary function from the system $\phi_i, \psi_j, \ldots$. Suppose x_1 takes on the value x_1^0,

$$x_2 \text{ the value } x_2^0, \ldots, z_1 \text{ the value } z_1^0, \ldots, z_{n_3} \text{ the value } z_{n_3}^0, \ldots,$$

where

$$x_1^0, x_2^0, \ldots, z_1^0, \ldots, z_{n_3}^0, \ldots$$

belong to the field M'. Each of these elements occurs in one of the summands

M'_i. We shall assume that

$$x_1^0 \subset M'_{k_1}; \quad x_2^0 \subset M'_{k_2}; \quad \ldots; \quad z_1^0 \subset M'_{g_1}; \quad \ldots.$$

We take from among all the sets $M'_{k_1}, \ldots, M'_{g_1}, \ldots$ that one which has the largest index. Suppose it is M'_t. The sets M'_n are defined so that each of the following ones contains all the elements of the preceding one. Therefore all the elements $x_1^0, \ldots, x_{n_1}^0, z_1^0, \ldots, z_{n_3}^0, \ldots$ occur in M'_t. But in this case $\theta(x_1^0, \ldots)$ is an element of the set M'_{t+1} and therefore it also occurs in M'.

We again consider formula (2), but we now assume that it is referred to the field M'. It is entirely possible to do this because the field M' is a part of M. The predicates $A_1^0, \ldots, A_q^0$, defined on M, are also defined for all elements of M', and the objects $a_1^0, \ldots, a_p^0$ belong to M' by construction. The decision functions

$$\ldots, \phi_i(x_1, \ldots, x_{n_1}), \ldots, \psi_j(x_1, \ldots, x_{n_1}, z_1, \ldots, z_{n_3}), \ldots$$

take on values from M' if their arguments belong to M'. Therefore the identical truth of the predicate

$$B(x_1, \ldots, x_{n_1}, \phi_1(x_1, \ldots, x_{n_1}), \ldots)$$

on the field M' is also a necessary and sufficient condition for the truth of formula (2) on the field M'. Since this predicate is actually true on M', we can conclude that formula (2) is true for the field M'. In this case, formula (1) is satisfiable on M'.

We have thus proved the assertion that if formula (1) is satisfiable on some arbitrary field, then it is satisfiable on a finite or denumerable part of it, which constitutes the Löwenheim theorem.

Axiomatic Characterization of Infinite Sets. We saw that if the axioms contain only predicates of one variable, then they cannot characterize an infinite field because such a system of axioms will always be satisfied in a finite field.

Axioms with predicates which depend on two or a greater number of variables can characterize infinite fields, and the question arises: Can they characterize non-countable fields? Suppose

$$A_1, A_2, \ldots, A_n$$

is a system of axioms containing only symbols of individual predicates. As we have already indicated above, we can always replace this system by one axiom:

$$A_1 \mathbin{\&} A_2 \mathbin{\&} \ldots \mathbin{\&} A_n$$

and always follow a line of reasoning about one axiom. In the case under consideration, the concept of the interpretability of axioms coincides with the concept of the satisfiability of this formula on some field (see §10). By virtue of the Löwenheim theorem, however, if this formula is satisfiable on some field, then it is also satisfiable on a finite or denumerable field. It

follows from this that it is impossible by means of axioms of the type under consideration to distinguish a non-denumerable field from either a denumerable or finite field.

It turns out, however, that axioms containing symbols for predicate variables can characterize non-denumerable sets and even sets of any prescribed power in the sense that all fields satisfied by them are non-denumerable or have precisely the prescribed power.

CHAPTER IV

PREDICATE CALCULUS

In the present chapter we shall give an axiomatic description of predicate logic which was considered in the preceding chapter from an informal point of view. We note that in contrast to propositional algebra, predicate logic is of an explicitly non-constructive character. All its concepts are defined for an arbitrary field or set of objects. In view of this, in the informal discussion of predicate logic, we had to depend on non-constructive principles of the theory of sets. The description of predicate logic which we shall give in the present chapter completely satisfies the requirements of Hilbert's finitism.

Predicate calculus, as is the case for every axiomatic system, contains symbols from which the formulae are constructed. Subsequently, from among all formulae, we isolate the so-called true formulae. The isolation of the true formulae in predicate calculus, as was the case in propositional calculus, is realized by pointing out a finite aggregate of formulae which are called axioms and by indicating the deduction rules which enable one to obtain, from the true formulae, new true formulae.

§1. Formulae of the predicate calculus

Every formula of the predicate calculus is a finite sequence of symbols of this calculus. Firstly we describe the symbols of the predicate calculus.

(1) Lower-case Latin letters with or without indices:

$$a, b, c, \ldots, x, y, z, \ldots, a_1, a_2, \ldots, x_1, x_2, \ldots.$$

These symbols are called *object variables.*

(2) Upper-case Latin letters with or without subscripts:

$$A, B, \ldots, A_1, A_2, \ldots$$

are called *propositional variables.*

(3) Upper-case Latin letters with superscripts:

$$F^p, G^p, \ldots, S^p, T^p, \ldots$$

and the same symbols with subscripts:

$$F_1^p, F_2^p, \ldots, G_1^p, G_2^p, \ldots, S_1^p, S_2^p, \ldots.$$

These symbols are called *predicate variables of p arguments* ($p = 1, 2, \ldots$).

(4) Symbols of the propositional calculus:

$$\&, \vee, \rightarrow, -.$$

(5) Brackets ( ).

(6) The symbol $\exists$.

DEFINITION OF FORMULA

As we have already pointed out, every formula is a finite sequence, or a row, of symbols. However, this still does not define the concept of formula. We must determine which rows of symbols are called formulae.

1. Every proposition variable is a formula.

2. If F^p is the symbol for a predicate variable and $a_1, a_2, \ldots, a_p$ are symbols for object variables, where the a_i are not necessarily distinct, then the row

$$F^p(a_1, a_2, \ldots, a_p)$$

is a formula. For such formulae we retain the terminology *predicate variable.* The formulae defined in 1 and 2 will be called *elementary formulae.*

Next, in the definition of a formula, we must distinguish the object variables occurring in the formula. Certain of them will be called *bound* and the others *free.* This distinction will be established parallel to the definition of a formula. In the elementary formulae, all object variables are free.

3. Suppose the formula A contains the free variable x. Then the rows

$$(x)A \quad \text{and} \quad (\exists x)A \tag{1}$$

are also formulae. The symbols (x) and $(\exists x)$ are called quantifiers: the first is the *universal quantifier* and the second the *existential quantifier.* The variable x in formulae (1) is called a *bound variable.* In particular, we shall say that in the formula $(x)A$ the variable x is bound by the quantifier (x) and in the formula $(\exists x)A$ the variable x is bound by the quantifier $(\exists x)$. And the remaining object variables, which are free in the formula A, also remain free in both formulae (1). Variables, which are bound in formula A, remain bound in formulae (1).

4. Suppose A and B are formulae such that there are no object variables which are bound in one formula and free in the other. Then the rows

$$(A \,\&\, B); \quad (A \vee B); \quad (A \rightarrow B); \quad -A \tag{2}$$

are formulae. In this connection, the free variables in formulae A and B remain free in all the formulae (2) and the bound variables in the formulae A and B remain bound in the formulae (2).

The definition of a formula in the predicate calculus has the same inductive character as the definition of a formula in the propositional calculus. It can be expressed as follows.

A formula is a row of symbols which can be constructed, starting from elementary formulae, by operations of transition from the formula A to the

formulae $(x)A$ *and* $(\exists x)A$, *from the formulae* A *and* B *to the formulae* $(A \,\&\, B)$, $(A \vee B)$, $(A \rightarrow B)$, *and* $-A$.

The principle of complete induction can be applied to prove any assertion about formulae. Such a proof has the following form. The assertion is proved for elementary formulae. After that, it is proved that from the assumption that this assertion is true for A follows its truth for $(x)A$ and $(\exists x)A$, and that its truth for A and B implies its truth for formulae (2). From this we conclude that the assertion is true for any formula.

It is clear from subsections 1-4, above, that all formulae of the propositional calculus are also formulae of the predicate calculus. In fact, among the formulae of the predicate calculus there are propositional variables and, taking them as our point of departure, we can construct formulae using the same operations as in the propositional calculus. The restrictions indicated in subsection 4 do not apply to formulae constructed from propositional variables only (i.e. not involving predicate symbols) inasmuch as they apply only to formulae which contain object variables.

Examples of Formulae

1. $(\exists x)(F^1(x) \rightarrow (y)G^2(y, z))$.

This row of symbols is a formula. In fact, $G^2(y, z)$ is a predicate variable which contains two free variables y, z, i.e. it is an elementary formula. According to subsection 3, the row $(y)G^2(y, z)$ is also a formula which contains the free variable z and the variable y bound by the universal quantifier. In the formulae $F^1(x)$ and $G^2(y, z)$ there are no variables which are bound in one formula and free in the other; therefore, the row

$$(F^1(x) \rightarrow (y)G^2(y, z))$$

is a formula, in virtue of subsection 4, in which x and z are free variables and y is bound by the quantifier (y). Finally, in virtue of subsection 3, the row

$$(\exists x)[F^1(x) \rightarrow (y)G^2(y, z)]$$

is also a formula in which the variable x is bound by the quantifier $(\exists x)$, the variable y is bound by the quantifier (y), and the variable z is free.

2. $((x)F^1(x) \vee (x)(\exists y)G^2(x, y))$.

It is easily seen that $(x)F^1(x)$ is a formula in which x is a bound variable and $(x)(\exists y)G^2(x, y)$ is a formula in which both variables are bound. These formulae satisfy the condition of subsection 4, above, inasmuch as all variables occurring in them are bound. Therefore, the given row is a formula.

3. $((\exists x)F^1(x) \,\&\, (\exists y)G^2(x, y))$.

This row is not a formula. Both of the rows $(\exists x)F^1(x)$ and $(\exists y)G^2(x, y)$ are formulae but the variable x is bound in the first whereas it is free in the second. Consequently, these formulae do not satisfy the condition of sub-

section 4, and combining them by the connective & does not produce a formula.

As in the propositional calculus, one can define the concept of *component of a formula* for formulae of the predicate calculus.

A component of an elementary formula is this formula itself.

The components of the formula $(x)A$ (or of $(\exists x)A$) are the formula itself and every component of the formula A.

The components of the formulae $(A \,\&\, B)$, $(A \vee B)$ and $(A \rightarrow B)$ are these formulae themselves and all components of the formulae A and B.

The components of the formula $— A$ are this formula itself and all components of the formula A.

We now introduce certain modifications in writing down formulae. Firstly, we shall omit certain brackets. The rules for omitting brackets are the same here as in the propositional calculus. Secondly, exterior brackets are omitted. For example, we shall write the formula

$$(A \,\&\, F^1(x))$$

in the form

$$A \,\&\, F^1(x),$$

the formula

$$((x)G^1(x) \vee A)$$

in the form

$$(x)G^1(x) \vee A,$$

and so forth. Furthermore, we shall omit brackets—taking into consideration the rule asserting that & is a stronger connective than $\vee$ and $\rightarrow$ and that $\vee$ is a stronger connective than $\rightarrow$. For example, we shall write the formula

$$((A \,\&\, B) \vee C)$$

in the form

$$A \,\&\, B \vee C,$$

and the formula

$$(B \rightarrow (F^1(x) \vee G^1(y)))$$

will be written in the form

$$B \rightarrow F^1(x) \vee G^1(y).$$

The negation sign will be placed over the formula, i.e. instead of $— A$ we shall write $\bar{A}$. In this connection, if A has exterior brackets, then we shall omit them in the formula $\bar{A}$.

Finally, in writing the predicate $F^p(x_1, \ldots, x_p)$, we shall omit the index p and we shall place the negation sign which applies to the predicate over the predicate symbol; for instance, $\bar{A}(x, y)$.

We shall agree that the letters in a formula which represent predicates with different numbers of variables are distinct. Then, in writing a formula, it is impossible to confuse a predicate with a propositional variable inasmuch as in a formula a predicate always occurs in a row of the form $F^p(x_1, \ldots, x_p)$, i.e. together with object variables which never happens in the case of a propositional variable.

EXAMPLES

1. $(-(y)(x)A^2(x, y)) \vee (C \,\&\, -(\exists y)B^1(y))$
can be written in the form

$$\overline{(y)(x)A(x, y)} \vee C \,\&\, \overline{(\exists y)B(y)}.$$

2. $-(x)-(\exists y)F^2(x, y)$
can be written as

$$\overline{(x)\overline{(\exists y)F(x, y)}}.$$

3. $((\exists x)(y)-((z)(H^1(x) \rightarrow G^2(y, z))))$
is written as

$$(\exists x)(y)\overline{(z)(H(x) \rightarrow G(y, z))}.$$

As in the case of the propositional calculus, we could give a definition of a formula out of which one could directly obtain the notation we arrived at after the indicated modifications. However, such a formulation would have created great inconvenience in the definition of a formula.

We introduce the concept of the domain of operation of a quantifier. Suppose a formula has the form: $(x)A$ or $(\exists x)A$. *Then the domain of operation of the quantifier* (x) [*respectively* $(\exists x)$] *is the formula* A.

It follows from the condition of subsection 4, above, regarding the non-coincidence of the notation for free and bound variables that:

(a) *the free and bound variables in a formula are denoted by different letters;*

(b) *if any quantifier occurs in the domain of operation of another quantifier, then the variables which are bound by these quantifiers are denoted by distinct letters.*

We shall prove assertions (a) and (b). Assertion (a) is obvious for elementary formulae inasmuch as they do not contain bound variables. We assume that (a) is valid for the formula $A(x)$ which contains the free variable x. Then (a) is also valid for the formulae $(x)A$ and $(\exists x)A$. In fact, all bound variables occurring in $(x)A$ and $(\exists x)A$, except x, are also bound in the formula A and therefore, by assumption, they differ from the free variables of the formula A. They are then, *a fortiori*, distinct from the free variables in $(x)A$ and $(\exists x)A$. And the variable x in these formulae differs from the free variables in these formulae inasmuch as in the formula A the variable x differs from the other free variables.

We now assume that (a) is valid for the formulae $\boldsymbol{A}$ and $\boldsymbol{B}$. Then (a) is valid for the formulae $\boldsymbol{A} \,\&\, \boldsymbol{B}$, $\boldsymbol{A} \vee \boldsymbol{B}$, $\boldsymbol{A} \rightarrow \boldsymbol{B}$ provided only that these expressions are formulae. In fact, in order that it be possible to construct these formulae, it is necessary that the free variables in A differ from the bound variables in $\boldsymbol{B}$, and conversely. But then if A and $\boldsymbol{B}$ did not contain the same free and bound variables, neither will these latter be contained in the newly created formulae $\boldsymbol{A} \,\&\, \boldsymbol{B}$, $\boldsymbol{A} \vee \boldsymbol{B}$ and $\boldsymbol{A} \rightarrow \boldsymbol{B}$. If assertion (a) is valid for $\boldsymbol{A}$, then it is obviously valid for $\bar{\boldsymbol{A}}$ also.

Thus, we have shown by induction the validity of assertion (a).

We shall prove assertion (b). It is obviously valid for elementary formulae.

We now assume that (b) is valid for the formula A which contains the free variable x. In virtue of the induction hypothesis, if one quantifier in A occurs in the domain of operation of the second quantifier, then the variables bound by these quantifiers are distinct. In formulae $(x)A$ and $(\exists x)A$, there is still another quantifier in whose domain of operation all the remaining quantifiers occur. But the variable x differs from all the remaining bound variables in these formulae inasmuch as the x in A is free and, consequently, it coincides with none of the bound variables. If (b) is valid for the formulae A and B, then (b) is also valid for the formulae $A \& B$, $A \vee B$ and $A \to B$, for, if one quantifier of such a formula occurs in the domain of operation of the other, then both these quantifiers belong to one of the formulae A or B and, in virtue of the induction hypothesis, the variables bound by these quantifiers are distinct. Assertion (b) has thus been proved.

Thus in order that a row of symbols in the predicate calculus be a formula, it is necessary that properties (a) and (b) be maintained. We shall call a failure of these conditions *collision of the variables.*

EXAMPLE

$$(x)[F(x) \to (\exists x)G(x, y)].$$

This row is not a formula inasmuch as (b) is not satisfied for it. It is easily seen that it cannot be constructed by operations 3 and 4 from elementary formulae. In fact, it is impossible to construct the formula $F(x) \to (\exists x)G(x, y)$ from the formulae $F(x)$ and $(\exists x)G(x, y)$ inasmuch as these formulae do not satisfy the condition of subsection 4, above. But the row

$$(x)F(x) \& (\exists x)G(x, y)$$

is a formula inasmuch as we do not require the variables which are bound by different quantifiers in the formulae A and B to be distinct in order that it be possible to form the formulae $A \& B$, $A \vee B$ and $A \to B$ from A and B.

§2. Change of variables in formulae

THEOREM. *If in a formula A we change the notation of the free as well as of the bound variables, replacing a letter by another everywhere where it occurs in such a way that conditions* (a) *and* (b) *are satisfied, then the row obtained in this manner will be a formula.*

This is true for elementary formulae, for instance, for $F(x, y, z)$, since, replacing the object variables occurring in the predicate in an arbitrary way, we again obtain a predicate, i.e. an elementary formula. We assume that the assertion is true for a formula A which contains the free variable x. We change the designation of the object variables in the formula $(x)A$ —not

disturbing conditions (a) and (b) in the process. If, in this connection, the letter x is replaced by a new letter, for example, by y, then y must differ from the variables by which all the remaining variables of formula $\boldsymbol{A}$ are replaced inasmuch as every variable, distinct from x, is either free in $(x)\boldsymbol{A}$ or it is bound by a quantifier occurring in the domain of operation of the quantifier (x). Since the renaming of the variables we have made in the formula $(x)\boldsymbol{A}$, and consequently in the formula $\boldsymbol{A}$ also, satisfies conditions (a) and (b), it follows, in virtue of the induction hypothesis, that the formula $\boldsymbol{A}'(y)$ which is obtained from $\boldsymbol{A}(x)$ as a result of this renaming of the variables is a formula. And since all the bound variables occurring in $\boldsymbol{A}'(y)$ are distinct from y, the row $(y)\boldsymbol{A}'(y)$ is a formula, which is what was to be proved.

It can be shown in the same way that if the assertion is true for $\boldsymbol{A}$, then it is also true for $(\exists x)\boldsymbol{A}$.

We assume that the assertion is true for the formulae $\boldsymbol{A}$ and $\boldsymbol{B}$ which are such that the bound variables of one formula are different from the free variables of the other formula. We shall prove that it is also true for the formula $\boldsymbol{A}\ \&\ \boldsymbol{B}$. We rename the variables of this formula so that conditions (a) and (b) are satisfied. After this, the components $\boldsymbol{A}$ and $\boldsymbol{B}$ of our formula transform into $\boldsymbol{A}'$ and $\boldsymbol{B}'$. In virtue of the induction hypothesis, the rows $\boldsymbol{A}'$ and $\boldsymbol{B}'$, which are obtained from $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively upon renaming the variables, satisfying conditions (a) and (b), are formulae. In virtue of condition (a), the bound variables in $\boldsymbol{A}'$ differ from the free variables in $\boldsymbol{B}'$, and conversely. Therefore, the row $\boldsymbol{A}'\ \&\ \boldsymbol{B}'$, formed from $\boldsymbol{A}'$ and $\boldsymbol{B}'$, is a formula, which is what we were required to prove.

In the same way, it can be proved that the assertion is true for the formulae $\boldsymbol{A} \vee \boldsymbol{B}$ and $\boldsymbol{A} \rightarrow \boldsymbol{B}$ if it is true for $\boldsymbol{A}$ and $\boldsymbol{B}$. Our assertion is obvious for the negation operation.

The theorem is thus proved for an arbitrary formula.

We note that the methods for renaming the object variables, pointed out above, do not exhaust all those for which the formula remains a formula. For example, in the formula

$$(x)[F(x) \vee G(y)]$$

renaming the letter y the letter x is not allowed by our requirements inasmuch as the variable y is free but x is bound. However, if such a replacement is performed, we obtain the row

$$(x)[F(x) \vee G(x)],$$

which is a formula. Therefore, conditions (a) and (b) are not necessary to ensure that as a result of renaming the variables a formula be again obtained from the initial formula. In the sequel, we shall need only those renamings of the variables which satisfy conditions (a) and (b), and we shall not search for a wider class.

§3. Axioms of the predicate calculus

We shall define true formulae in the same manner as that applied in the description of the propositional calculus. We shall consider certain well-defined formulae, prescribed and finite in number, to be true and we shall call them axioms of the predicate calculus. After that, we shall lay down the rules of formation of new true formulae from those which were already obtained or from the axioms. Those and only those formulae which can be obtained upon application of these rules, starting from axioms, will be considered true.

AXIOMS OF THE PREDICATE CALCULUS

I

1. $A \to (B \to A)$,
2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

II

1. $A \& B \to A$,
2. $A \& B \to B$,
3. $(A \to B) \to ((A \to C) \to (A \to B \& C))$.

III

1. $A \to A \vee B$,
2. $B \to A \vee B$,
3. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$.

IV

1. $(A \to B) \to (\bar{B} \to \bar{A})$,
2. $A \to \bar{\bar{A}}$,
3. $\bar{\bar{A}} \to A$.

V

1. $(x)F(x) \to F(y)$,
2. $F(y) \to (\exists x)F(x)$.

The first four groups of axioms are none other than the axioms of the propositional calculus (see page 48). To them we adjoin further two new axioms, comprising the group V. And this exhausts the set of axioms of the predicate calculus.

§4. Rules for the formation of true formulae

1. Rule of inference. *If G and $G \to H$ are true formulae, then H is also a true formula.*

This rule is formulated the same way as in the propositional calculus. It is only that the class of formulae, to which this rule is applied, is more extensive here.

2. Rule of substitution in a propositional variable and in a predicate variable. This rule is analogous to the rule for substitution which we had for the propositional calculus (see Chapter II). There, it reduced to the fact that upon replacing a variable proposition in a true formula by an arbitrary formula, we obtained a true formula. In predicate logic we shall deal with the replacement of propositional variables and of predicate variables by formulae. But whereas in the propositional calculus a formula, which replaces a propositional variable in a true formula, could be perfectly arbitrary, here it is necessary to place certain supplementary conditions on these formulae, since otherwise as a result of substitution one can obtain an expression which is not even a formula.

We shall first describe the replacement operations which we need for an arbitrary formula independent of whether it is true or not, for the time being in a non-rigorous but more graphic manner.

Replacement of a propositional variable. Suppose the formula $A(A)$ contains the propositional variable A. Then we can replace in the formula A the letter A, wherever it occurs, by an arbitrary formula G which satisfies the following conditions.

(b^1) *The free variables in G are denoted by letters which are distinct from the bound variables in A and the bound variables in G are denoted by letters which are distinct from the free variables in A.*

(b^2) *If A, in A, occurs in the domain of operation of a quantifier which is denoted by some letter, then this letter does not occur in G.*

Such a replacement is called *the substitution of the formula G in the variable A.*

EXAMPLE. Suppose $A(A)$ is the formula

$$(x)(y)[A \vee (z)H(z, x) \;\&\; (\bar{A} \vee F(x, y))].$$

In this case, it is impossible to replace A by the formula $(x)B(x)$ or the formula $(\exists x)C(x)$ inasmuch as condition (b^2) is not satisfied in such a replacement. But if we none the less perform this replacement, then the row obtained will not be a formula inasmuch as it contains two quantifiers, one of which occurs in the domain of operation of the other, which are denoted by the same letter. Replacement of the letter A by the formula

$$(z)(t)[A \;\&\; H(z, t) \vee B \;\&\; F(z, t)]$$

is possible because in this case conditions (b^1), (b^2) are satisfied. The row obtained as a result of such a replacement:

$$(x)(y)\{(z)(t)(A \;\&\; H(z, t) \vee B \;\&\; F(z, t)) \vee (z)H(z, x) \;\&$$
$$\&\; [\overline{(z)(t)(A \;\&\; H(z, t) \vee B \;\&\; F(z, t))} \vee F(x, y)]\}$$

is a formula.

Replacement of a predicate variable. Suppose the formula $A(F)$ contains the predicate variable F of n arguments and suppose we have the formula

$\boldsymbol{B}(t_1, t_2, \ldots, t_n)$, containing n free variables $t_1, t_2, \ldots, t_n$ (generally speaking, $\boldsymbol{B}$ can contain other variables also), where $t_1, t_2, \ldots, t_n$ are letters which are different from all the object variables of formula $\boldsymbol{A}$. If, for the formula $\boldsymbol{B}$, there hold condition (b^1) and the further condition:

(b_2^2). *If F, in $\boldsymbol{A}$, occurs in the domain of operation of a quantifier which binds some letter, and this letter does not occur in $\boldsymbol{B}$*, then it is possible to substitute the formula $\boldsymbol{B}$ in A in place of the predicate F. The operation of substitution of the formula $\boldsymbol{B}(t_1, \ldots, t_n)$ into the formula $A(F)$ in place of $F(\ldots)$ signifies the replacement of every elementary formula of the form $F(x_1, x_2, \ldots, x_n)$ (where $x_1, x_2, \ldots, x_n$ are certain variables which are not necessarily distinct), occurring in $A(F)$, by an expression obtained from $\boldsymbol{B}$ by replacing the variables $t_1, t_2, \ldots, t_n$ by the letters $x_1, x_2, \ldots, x_n$ respectively. In this connection, it must be very clearly pointed out to which of the variables $t_1, t_2, \ldots, t_n$ each of the empty places in $F(\ldots)$ corresponds.

EXAMPLE. Suppose the formula A has the form:

$$(x)(\exists y)(\exists z)[F(x, y) \vee \overline{F}(x, z)].$$

It is required to perform a substitution, replacing F by the formula

$$(u)(\exists v)[H(u, t_1) \vee H(v, t_2)].$$

Suppose, in this connection, that the first empty place in $F(\ldots)$ corresponds to the variable t_1 and the second to t_2. Conditions (b^1) and (b_2^2) are satisfied here and the result of the substitution will be the formula

$$(x)(\exists y)(\exists z)\{(u)(\exists v)[H(u, x) \vee H(v, y)] \vee \\ \vee \overline{(u)(\exists v)[H(u, x) \vee H(v, z)]}\}.$$

It is easily seen that if conditions (b^1) and (b^2) or (b_2^2) are not satisfied, then, generally speaking, as a result of the replacement we shall obtain expressions which are not formulae.

EXAMPLES.

1. It is required to substitute $U(x)$ in place of A in the formula

$$A \vee (x)F(x).$$

We obtain the row

$$U(x) \vee (x)F(x).$$

This row is not a formula in virtue of the collision of variables which arises; here, condition (b^1) is not satisfied.

2. It is required to substitute $(x)U(x)$ in place of A in the formula

$$(x)(A \rightarrow F(x)).$$

We obtain the row

$$(x)((x)U(x) \rightarrow F(x)),$$

which is not a formula. In the substitution under consideration, condition (b^2) is not maintained.

Another definition of the operation of substitution. The operation of substitution will be represented by one of the symbols

$$R_A^B,\ R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)},$$

placed before the formula on which the operation of substitution is performed —for example,

$$R_A^B(A),\ R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(A)$$

where A represents a propositional variable and F is a predicate variable of n arguments. Among its free object variables, the formula $B(t_1, \dots, t_n)$ contains the specially specified variables $t_1, \dots, t_n$, the number of which is equal to the number of variables in the predicate F, i.e. n.

If the formula $\mathbf{A}$ *contains the propositional variable* A (*respectively, the predicate variable* F), *then the substitution operation*

$$R_A^B(\mathbf{A}),\ \text{or}\ R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A})$$

is applicable if for the formulae $\mathbf{B}$ *and* $\mathbf{A}$ (*or for* $\mathbf{B}(t_1, \dots, t_n)$ and $\mathbf{A}$) *conditions* (b^1) *and* (b^2) (*or* (b^1) *and* (b_2^2)) *are satisfied and, if, in addition, the specified variables* $t_1, \dots, t_n$ *do not occur in the formula* $\mathbf{A}$.

We give an inductive definition of the substitution operation, starting with the elementary formulae:

$$R_A^B(A) \text{ is } \mathbf{B};$$

$$R_A^B(C) \text{ is } C,$$

where C is a propositional variable distinct from A;

$$R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(F(x_1, \dots, x_n)) \text{ is } \mathbf{B}(x_1, \dots, x_n);$$

$$R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(G(x_1, \dots, x_n)) \text{ is } G(x_1, \dots, x_n),$$

if $G(x_1, \dots, x_n)$ is a predicate variable, distinct from $F(x_1, \dots, x_n)$.

We shall assume that the substitution operation is defined for the formulae $\mathbf{A}_1$ and $\mathbf{A}_2$. Then the substitution operation for the formulae $\mathbf{A}_1 \mathbin{\&} \mathbf{A}_2$, $\mathbf{A}_1 \vee \mathbf{A}_2$, $\mathbf{A} \to \mathbf{A}_2$ and $\bar{\mathbf{A}}$ is defined in the following way:

$$R_A^B(\mathbf{A}_1 \mathbin{\&} \mathbf{A}_2) \text{ is } R_A^B(\mathbf{A}_1) \mathbin{\&} R_A^B(\mathbf{A}_2);$$

$$R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A}_1 \mathbin{\&} \mathbf{A}_2) \text{ is } R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A}_1) \mathbin{\&} R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A}_2);$$

$$R_A^B(\mathbf{A}_1 \vee \mathbf{A}_2) \text{ is } R_A^B(\mathbf{A}_1) \vee R_A^B(\mathbf{A}_2);$$

$$R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A}_1 \vee \mathbf{A}_2) \text{ is } R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A}_1) \vee R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A}_2);$$

$$R_A^B(\mathbf{A}_1 \to \mathbf{A}_2) \text{ is } R_A^B(\mathbf{A}_1) \to R_A^B(\mathbf{A}_2);$$

$$R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A}_1 \to \mathbf{A}_2) \text{ is } R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A}_1) \to R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A}_2);$$

$$R_A^B(\bar{\mathbf{A}}) \text{ is } \overline{R_A^B(\mathbf{A})};$$

$$R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\bar{\mathbf{A}}) \text{ is } \overline{R_F^{B\,(t_1,\ \dots,\ t_n)}_{\ \ (\dots)}(\mathbf{A})}.$$

Suppose the operations $R_A^B(A(x))$ and $R_{F(\dots)}^{B(t_1,\dots,t_n)}(A(x))$ have been defined. Then

$$R_A^B((x)A(x)) \text{ is } (x)R_A^B(A(x))$$

and

$$R_{F(\dots)}^{B(t_1,\dots,t_n)}((x)A(x)) \text{ is } (x)R_{F(\dots)}^{B(t_1,\dots,t_n)}(A(x)).$$

In an analogous manner,

$$R_A^B((\exists x)A(x)) \text{ is } (\exists x)R_A^B(A(x))$$

and

$$R_{F(\dots)}^{B(t_1,\dots,t_n)}((\exists x)A(x)) \text{ is } (\exists x)R_{F(\dots)}^{B(t_1,\dots,t_n)}(A(x)).$$

The substitution operation is completely defined by these relations.

It is not difficult to prove that if the substitution operation is applied to a formula maintaining all the conditions indicated above, then a formula is obtained as a result. We shall not consider this point further.

3. The rule for the replacement of a free object variable. Suppose the formula A is a true formula in the predicate calculus and that the formula A' is obtained from A by the replacement of an arbitrary free object variable by another free object variable such that the variable being replaced is replaced in the same way everywhere it occurs in the formula A; then A' is a true formula in the predicate calculus.

Of course, we assume that the indicated replacement of variables does not lead to a collision of variables. This already follows from the assumption that A' must also be a formula.

EXAMPLE. Let us consider the formula

$$(x)(F(x) \to G(y)) \,\&\, (F(y) \to (\exists x)G(x)).$$

We perform a substitution in it by replacing the variable y by the variable z; we obtain the formula

$$(x)(F(x) \to G(z)) \,\&\, (F(z) \to (\exists x)G(x)).$$

As is required, we changed the variable y everywhere where it occurs. We note that it is impossible to replace the variable y by the variable x because, according to the definition, the free variable y must be replaced by a free variable only.

4. The rule for renaming bound object variables. If the formula A is a true formula of the predicate calculus, then the formula A' obtained from A by a replacement of the bound variables by other bound variables, which are distinct from all the free variables in the formula A, is also a true formula. In this connection, the replaced bound variable in the formula A must be replaced in the same manner everywhere in the domain of operation of the quantifier which binds the given variable and in the same quantifier.

The operation of renaming bound variables differs essentially from the operation of substitution in free variables. In renaming bound variables,

we are not obliged to rename them wherever they occur in the formula A but only in the domain of operation of the quantifier which binds the given variable. This means that identical variables, for which the quantifiers binding them have non-overlapping domains of operation, can be renamed separately or one of them can be renamed and the other not.

EXAMPLES:

1. Let us consider the formula

$$(\exists x)F(x) \to (x)G(x).$$

Renaming the bound variables, we can obtain the following formula from the given formula:

$$(\exists y)F(y) \to (z)G(z).$$

Here we replaced the variable x in the domain of operation of one quantifier by the variable y and by the variable z in the domain of operation of the other quantifier.

Inasmuch as the domains of operation of these quantifiers do not overlap, we have accomplished a renaming of the bound variables in a perfectly legitimate manner. Renaming the bound variables differently, we can obtain the following formula from the same formula:

$$(\exists x)F(x) \to (z)G(z).$$

2. We consider the formula

$$(\exists v)(\exists x)(y)[(F(x, y) \vee (\exists z)G(z)) \,\&\, G(y) \vee H(v)].$$

When renaming the variable y in this formula, we ought to replace it in the same way everywhere where it occurs. The renaming leading to the formula

$$(\exists v)(\exists x)(u)[(F(x, u) \vee (\exists z)G(z)) \,\&\, G(u) \vee H(v)]$$

is completely legitimate, but the replacement of the variable y leading to the formula

$$(\exists v)(\exists x)(u)[(F(x, u) \vee (\exists z)G(z)) \,\&\, G(v) \vee H(v)]$$

is not a legitimate renaming of the bound variables inasmuch as in this case we renamed the variable y in the domain of operation of the same quantifier (y) differently—replacing it in one place by the variable u and by the variable v in another place.

5. Rules for binding by a quantifier. *First rule for binding by a quantifier:* If $B \to A(x)$ is a true formula and B does not contain the variable x, then $B \to (x)A(x)$ is also a true formula.

Second rule for binding by a quantifier: If $A(x) \to B$ is a true formula and B does not contain the variable x, then $(\exists x)A(x) \to B$ is also a true formula.

[In the formulation of the first and second rules for binding by a quantifier, we are dealing with a well-defined object variable x. However, by

means of the rules for substitution in a free object variable and the renaming of bound variables, it is easily generalized to an arbitrary object variable.]

We note that among the true formulae of the predicate calculus one finds all true formulae of the propositional calculus. In fact, the predicate calculus contains among its axioms all axioms of the propositional calculus, and among the rules for the formation of true formulae both rules of the propositional calculus: the inference rule and the substitution rule. In applications to formulae of the propositional calculus, these two rules of the predicate calculus coincide with the corresponding rules of the propositional calculus (see Chapter II). Thus, applying these rules to the axioms, we can obtain all the true formulae of the propositional calculus. The question whether there exist formulae in the propositional calculus which are true in the predicate calculus but not true in the propositional calculus is answered in the negative. This follows easily from the consistency of the predicate calculus and the completeness of the propositional calculus in the restr icted sense. We shall talk about this in the following section.

We pose the problem of the relation between the concept of a true formula in the calculus just described and the informal concept of an identically true formula considered in Chapter III. It is easy to see that every true formula in the predicate calculus is at the same time an identically true formula in the naïve set-theoretic sense. Firstly, it is manifest that the axioms of the predicate calculus are identically true. Secondly, the application of the rule of inference of the predicate calculus to identically true formulae leads to identically true formulae. This is obvious for the deduction rule. It is also obvious for the rules of substitution for free and bound variables.

We consider the substitution rule. $R_{F(\ldots)}^{B(t_1,\ldots,t_n)}(A)$ is the result of replacing, in the formula A, the elementary predicate $F(\ldots)$ by the formula $B(t_1, \ldots, t_n)$ everywhere where it occurs in A. At the same time, each of the variables $t_1, \ldots, t_n$ is replaced by the corresponding variable in the symbol $F(\ldots)$ being replaced. But the formula $B(t_1, \ldots, t_n)$ can also be considered as a predicate of n variables if one fixes the values of all the remaining free variables. Since, by hypothesis, formula A is true for every field and for arbitrary replacements of the predicate variables by individual predicates, it is obvious that it is also true upon replacement of F by the formula $B(t_1, \ldots, t_n)$. The case of the rule of substitution in a proposition variable is evident.

We consider the rule for binding by a quantifier. Let

$$B \to A(x) \tag{1}$$

be an identically true formula and suppose B does not contain the variable x. Then, in the informal sense, B does not depend on x for an arbitrary field and any predicates. Therefore, if the formula B turned out to be true upon some replacement of the variables, then it is true for every value of x for the

given replacements of the remaining variables. Since (1) is true by hypothesis, $\boldsymbol{A}(x)$ is also true for arbitrary x and the given replacements of the remaining variables. But then the formula $(x)\boldsymbol{A}(x)$ and consequently the formula

$$\boldsymbol{B} \to (x)\boldsymbol{A}(x) \tag{2}$$

are also true for the given replacements of the remaining variables. But if for some replacement of the variables the formula $\boldsymbol{B}$ is false, then formula (2) is still true. Thus, formula (2) is an identically true formula which is what we required to prove.

It is proved in the same way that the application to an identically true formula of the second rule for binding by a quantifier leads to an identically true formula.

We have thus shown that every formula deduced from the axioms by the rules of the predicate calculus is an identically true formula in the informal sense. We note that at the same time we obtain a proof of the consistency of the predicate calculus on the basis of the naïve theory of sets.

In fact, if every formula proved in the predicate calculus is identically true, then two formulae of which one is the negation of the other cannot both be provable because they cannot both be identically true simultaneously. However, this proof of consistency rests on the concept of an actual infinity. It cannot be utilized in the solution of the problem of the consistency of the theory of sets itself because this would lead to a circular argument. Moreover, a rigorous proof of the consistency of the predicate calculus which we shall consider in the following section is based on the same idea as that introduced here.

§5. Consistency of the predicate calculus

The question of the consistency of the predicate calculus is easily solved in the positive sense. The formulation of the problem of the consistency of the predicate calculus is the same as for the propositional calculus: a calculus is said to be intrinsically inconsistent provided some formula together with its negation is provable in it.

For the predicate calculus and for an arbitrary logical system obtained from the predicate calculus by the adjunction of new formulae in the role of axioms, as well as for the propositional calculus, one can assert that if this system were inconsistent, then an arbitrary formula would be true in it. In fact, let us assume that in the predicate calculus we have proved the formula $\boldsymbol{A}$ as well as the formula $\bar{\boldsymbol{A}}$. The formula

$$A \mathbin{\&} \bar{A} \to B$$

is a true formula in the predicate calculus, inasmuch as it is true in the propositional calculus. Performing a substitution and replacing the letter A by the formula $\boldsymbol{A}$, we obtain the true formula:

$$\boldsymbol{A} \mathbin{\&} \bar{\boldsymbol{A}} \to B.$$

On the basis of the rule of inference, we assert that B is a true formula. Performing a substitution in B of an arbitrary formula $\boldsymbol{B}$, we find that $\boldsymbol{B}$ is a true formula. Thus, also for the predicate calculus, finding some non-true formula is a proof of its consistency.

The formal meaning of the proof of consistency consists in the following. We shall consider formulae informally and understand them as was done in the preceding chapter. Namely, we shall assume that all predicates occurring in the formulae are defined on some field M. If this field consists of one element then the quantifiers can be discarded because both of the propositions $(x)A(x)$ and $(\exists x)A(x)$ for a field consisting of one element a are equivalent to the proposition $A(a)$. Thus, for this interpretation all formulae of the predicate calculus are replaced by formulae of the propositional calculus. At the same time, all axioms of the predicate calculus become true formulae of the propositional calculus and the rules for the formation of true formulae transform into rules of the propositional calculus—basic or deducible. If a formula which is a letter A were provable in the predicate calculus, then in the transformed system it would also be provable. But then the transformed system would be inconsistent. But the transformed system is the propositional calculus which, as is known (see Chapter II), is consistent. After these preliminary remarks, we shall give a formal proof of the consistency of the predicate calculus.

We shall set into correspondence with each formula of the predicate calculus a formula A^* according to the following law. To a propositional variable we assign the same propositional variable.

The letter F is assigned to an elementary formula of the form

$$F(x, y, \ldots, u).$$

If to the formulae A_1, A_2, A, $B(x)$ are assigned the formulae A_1^*, A_2^*, A^*, B^*, then to the formulae

$$(1)\ A_1 \,\&\, A_2; \quad (2)\ A_1 \vee A_2; \quad (3)\ A_1 \rightarrow A_2;$$
$$(4)\ \bar{A}; \quad (5)\ (x)B(x); \quad (6)\ (\exists x)B(x)$$

are assigned the formulae

$$(1')\ A_1^* \,\&\, A_2^*; \quad (2')\ A_1^* \vee A_2^*; \quad (3')\ A_1^* \rightarrow A_2^*;$$
$$(4')\ \bar{A}^*; \quad (5')\ B^*; \quad (6')\ B^*.$$

We see that the presence of quantifiers in a formula of the predicate calculus has absolutely no influence on the formula which is set into correspondence with it. To the formulae $B(x)$, $(x)B(x)$ and $(\exists x)B(x)$ are assigned the same formula itself. We can briefly describe the formula A^* which is set into correspondence with the formula A in the following way: *The formula A^* is obtained from the formula A if in the latter we cancel all quantifiers and delete all object variables leaving only the letter F of each of the elementary*

predicates $F(x, y, \ldots, u)$ occurring in the formula. It follows from this law of correspondence that formulae which we set into correspondence with formulae of the predicate calculus are formulae of the propositional calculus. For elementary formulae of the predicate calculus this is immediately clear. Forming from elementary formulae an arbitrary formula of the predicate calculus, we apply operations 3 and 4, described in §1 (see page 133). But then the formula corresponding to it is formed from formulae, corresponding to elementary formulae, by means of operations 4, i.e. by the same operations except the operation of binding by a quantifier. But if we apply the operation of binding by a quantifier to any formula of the predicate calculus, then the formula corresponding to it remains unchanged. Thus, for every formula of the predicate calculus the formula corresponding to it is formed from variable propositions with the aid of the operations of the propositional calculus and, consequently, it is itself a formula of the propositional calculus.

We shall show that to true formulae of the predicate calculus there correspond true formulae in the propositional calculus. We shall give a proof by induction.

To axioms of the predicate calculus there correspond true formulae in the propositional calculus.

In fact, to the axioms of groups I-IV there correspond these axioms themselves. These formulae are also axioms of the propositional calculus. And to both axioms of group V there corresponds the formula $F \rightarrow F$ which is a true formula of the propositional calculus.

To rules for the formation of true formulae in the predicate calculus there correspond certain rules for the formation of new formulae. We shall show that these rules are the rules for the formation of true formulae in the propositional calculus. It will follow directly from this situation that formulae which correspond to true formulae of the predicate calculus are true formulae of the propositional calculus.

Rule of inference in the predicate calculus: if A and $A \rightarrow B$ are true formulae, then B is also a true formula. But if the corresponding formulae A^* and $A^* \rightarrow B^*$ are true formulae in the propositional calculus then the formula B^* is also a true formula in the propositional calculus because the rule of inference is in the propositional calculus also.

Rule for the substitution in a free object variable and the rule for the renaming of a bound variable. We note that if the formulae A and A' differ from each other only in the object variables, then the formulae corresponding to them coincide. This follows directly from the brief description of the corresponding formula which was made above. It follows from this that if, to the formula A in the predicate calculus, there corresponds the true formula A^* in the propositional calculus, then to the formula A' obtained from A by means of a renaming of the object variables, or a substitution in the free variables, there corresponds this same true formula A^*.

Substitution rule. We first note that if H is a formula obtained as the result of a substitution in which, in the formula A, the letter A or the predicate $F(\ldots)$ is replaced by the formula B, then the formula H^* corresponding to H is obtained as the result of a substitution in the formula A^*, by a replacement in A^* of the letter A or F by the formula B^*. Applying the symbol for the substitution operation, our assertion can be written thus:

$$[R_A^B(A)]^* \text{ is } R_A^{B^*}(A^*)$$

and

$$[R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A)]^* \text{ is } R_F^{B^*}(A^*).$$

[We note that the operation $R_F^{B^*}(A^*)$, which represents the substitution for a variable F in a formula A^* of the propositional calculus of another formula B^*, which also is a formula of the propositional calculus, can always be performed.]

This assertion, obviously, is valid if A is an elementary formula A or F.

We shall consider further the operation of forming new formulae and we shall prove by induction that if our assertion is valid for the formulae A_1, A_2, $A(x)$, then it is also valid for formulae obtained from these upon application of the logical operations of addition, multiplication, implication, negation and binding with a quantifier. This can be done quickly by making use of the property of the commutativity of the operations R_A^B and $R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}$ with these logical operations. We shall carry out the proof only for the substitution operator $R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}$ and the logical operations of multiplication and binding with a quantifier.

It is required to prove that the formula

$$[R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A_1 \mathbin{\&} A_2)]^*$$

coincides with

$$R_F^{B^*}(A_1^* \mathbin{\&} A_2^*)$$

if it is known that our assertion is valid for the formulae A_1 and A_2. By virtue of the properties of the operator $R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}$, we have:

$$[R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A_1 \mathbin{\&} A_2)]^*$$

is

$$[R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A_1) \mathbin{\&} R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A_2)]^*. \tag{1}$$

But, in virtue of the interrelationship between (1) and (1′) (see page 147),

$$[R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A_1) \mathbin{\&} R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A_2)]^*$$

is

$$[R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A_1)]^* \mathbin{\&} [R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A_2)]^*. \tag{2}$$

In virtue of the induction assumption,

$$[R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A_1)]^* \text{ is } R_F^{B^*}(A_1^*)$$

and

$$[R_F^{B}{}_{(\ldots)}^{(t_1,\ldots,t_n)}(A_2)]^* \text{ is } R_F^{B^*}(A_2^*). \tag{3}$$

It follows from (1), (2) and (3) that

$$[R_F^B{}^{(t_1, \ldots, t_n)}_{(\ldots)}(A_1 \,\&\, A_2)]^* \text{ is } R_F^{B^*}(A_1^*) \,\&\, R_F^{B^*}(A_2^*)$$

or, in virtue of the properties of the operator $R_F^{B^*}$,

$$[R_F^B{}^{(t_1, \ldots, t_n)}_{(\ldots)}(A_1 \,\&\, A_2)]^* \text{ is } R_F^{B^*}(A_1^* \,\&\, A_2^*).$$

We now consider the operation of binding with the quantifier (x). It is necessary to prove that

$$[R_F^B{}^{(t_1, \ldots, t_n)}_{(\ldots)}(x)A(x)]^* \text{ is } R_F^{B^*}[(x)A(x)]^*.$$

In fact,

$$[R_F^B{}^{(t_1, \ldots, t_n)}_{(\ldots)}(x)A(x)]^* \text{ is } [(x)R_F^B{}^{(t_1, \ldots, t_n)}_{(\ldots)}(A(x))]^*$$

or

$$[R_F^B{}^{(t_1, \ldots, t_n)}_{(\ldots)}(A(x))]^*.$$

In virtue of the induction hypothesis,

$$[R_F^B{}^{(t_1, \ldots, t_n)}_{(\ldots)}(A(x))]^* \text{ is } R_F^{B^*}(A^*)$$

or, what amounts to the same thing:

$$[R_F^B{}^{(t_1, \ldots, t_n)}_{(\ldots)}(A(x))]^* \text{ is } R_F^{B^*}[(x)A(x)]^*,$$

from which it follows that

$$[R_F^B{}^{(t_1, \ldots, t_n)}_{(\ldots)}((x)A(x))]^* \text{ is } R_F^{B^*}[(x)A(x)]^*,$$

and this is what we were required to prove.

Our assertion is proved in exactly the same way for the remaining logical operations.

Now let H be a true formula of the predicate calculus, where H is obtained from the true formula A as the result of substituting the formula $\boldsymbol{B}$ for $F(\ldots)$ (or for A). By assumption, to the formula A there corresponds the true formula A^* of the propositional calculus, and to the formula $\boldsymbol{B}$ the formula $\boldsymbol{B}^*$ of the propositional calculus. But then, according to what we have proved, to the formula H there corresponds the formula H^*, obtained as the result of a substitution in the true formula A^* of the propositional calculus, in place of F (or A), of the formula $\boldsymbol{B}^*$. Consequently, H^* is a true formula of the propositional calculus, which is what we required to prove.

Rule for binding by a quantifier. Suppose $\boldsymbol{B} \to A(x)$ is a true formula and that $\boldsymbol{B}$ does not contain the variable x. The formula $\boldsymbol{B}^* \to A^*$ corresponds to it. We shall assume that this formula is true. To the formula

$$\boldsymbol{B} \to (x)A(x) \tag{4}$$

there corresponds the formula

$$\boldsymbol{B}^* \to [(x)\, A\, (x)]^*,$$

or, what amounts to the same thing:

$$\boldsymbol{B}^* \to A^*.$$

Thus, to the formula (4) there corresponds the very same formula as corresponds to the formula $\boldsymbol{B} \rightarrow A(x)$, i.e. a true formula of the propositional calculus.

Our assertion is proved for the second rule for binding by a quantifier in exactly the same way.

We have thus shown that:

If to the formulae

$$\boldsymbol{A}, \boldsymbol{B}, \ldots \tag{5}$$

of the predicate calculus there correspond the true formulae

$$\boldsymbol{A}^*, \boldsymbol{B}^*, \ldots$$

of the propositional calculus, then to the formulae obtained from formulae (5) *upon application of the rules of inference, substitution, renaming of variables and binding by a quantifier, there correspond formulae which are true in the propositional calculus.*

Since to the axioms of the predicate calculus there correspond true formulae of the propositional calculus, it follows from this that to every true formula of the predicate calculus there corresponds a true formula of the propositional calculus.

The intrinsic consistency of the predicate calculus follows directly from this. In fact, if the predicate calculus were inconsistent, then every formula in it would be true. But then the formula corresponding to A, i.e. A itself, would be true in the propositional calculus. And this, as we know, is not true because the propositional calculus is consistent (if A were true, then every formula would be true inasmuch as every formula $\boldsymbol{B}$ of the propositional calculus can be obtained from A by a substitution of $\boldsymbol{B}$ in place of A).

We can now answer the question posed above: Can a formula of the propositional calculus which is not true in the propositional calculus be true in the predicate calculus? We shall prove that such a formula cannot exist. Let $\boldsymbol{A}$ be a formula of the propositional calculus which is true in the predicate calculus; then the formula $\boldsymbol{A}^*$ corresponding to it is true in the propositional calculus. But since $\boldsymbol{A}$ is itself a formula of the propositional calculus, it follows that $\boldsymbol{A}^*$ coincides with $\boldsymbol{A}$ and hence $\boldsymbol{A}$ is true in the propositional calculus, which is what we required to prove.

We have thus shown that every formula of the propositional calculus which is true in the predicate calculus is a true formula in the propositional calculus.

§6. Completeness in the restricted sense

The question of completeness in the wide and narrow senses also arises in relation to the predicate calculus (see Chapter II, §10). We shall consider the question of completeness in the wide sense in the sequel. But the

question of completeness in the narrow sense is easily solved negatively. We shall consider it immediately.

We first recall the definition of completeness in the narrow sense. A logical system L is said to be complete in the narrow sense if it is impossible without giving rise to a contradiction to adjoin to its axioms — in the role of a new axiom—any formula not deducible in L. In distinction to the propositional calculus, the predicate calculus turns out not to be complete in the narrow sense. One can adjoin to its axioms—without giving rise to a contradiction—the formula

$$(\exists x)F(x) \rightarrow (x)F(x) \tag{1}$$

which is not provable in it.

The proof of this fact is realized on the basis of the same correspondence, in virtue of which we associated with each formula A of the predicate calculus the formula A^* of the propositional calculus. It follows from the line of reasoning pursued in §5 that every formula to which there corresponds a true formula of the propositional calculus can be adjoined—without giving rise to a contradiction—to the axioms of the predicate calculus. To formula (1) there corresponds the formula $F \rightarrow F$ in the propositional calculus, which formula is true in the propositional calculus.

Thus, formula (1) can be adjoined to the axioms of the predicate calculus. It may appear strange that one can adjoin such a formula, which is clearly invalid, to the axioms of the predicate calculus. In order to clarify this problem, we turn to the informal meaning of the formulae of the predicate calculus. The situation is this: nothing follows from the general logical axioms as to what objects and how many of them exist in the field M to which our propositions and predicates are referred. From general logical propositions it is impossible to conclude, for instance, that the field M contains more than one element. But if the field M contains only one element, then formula (1) is true for it. Furthermore, our method of proof of the consistency of these or other axioms consisted in this that we interpreted all our formulae on a field consisting of one element.

In order to prove that the predicate calculus is not complete in the narrow sense it is still necessary for us to show that formula (1) is not deducible from the axioms of the predicate calculus. From the informal viewpoint, this problem is perfectly clear. For, from the universal truth of formula (1) the impossibility of the existence of more than one element in the field would follow. And if it is impossible to prove the existence of more than one object from general logical propositions, then the existence of only one object cannot be proved also.

However, it is possible to give a completely rigorous proof of the fact that formula (1) cannot be formally deduced from the axioms of the predicate calculus. We shall not carry out this proof in detail—rather, we shall limit

ourselves to a discussion of the basic idea. This idea consists in the use of an interpretation of the formulae of the predicate calculus in a field consisting of two elements which one can take to be the numbers 1 and 2. We set into correspondence with each formula a formula in which the operations of binding by a quantifier is replaced in the following manner:

$$(x)A(x) \text{ is replaced by } A(1) \,\&\, A(2),$$

$$(\exists x)A(x) \text{ is replaced by } A(1) \vee A(2).$$

We shall call a formula in the predicate calculus legitimate if for arbitrary replacements of the free variables by the numbers 1 and 2 it is a true formula of the propositional calculus. To each formula of the predicate calculus one can set into correspondence the formula A^{**} which is a true formula in the propositional calculus.

This can be verified directly for axioms. Axioms of the groups I-IV contain neither variables nor quantifiers; therefore, the formulae corresponding to them are these axioms themselves, i.e. true formulae of the propositional calculus.

We consider axiom V.1:

$$(x)F(x) \rightarrow F(y).$$

Replacing the quantifier by the product, we obtain

$$F(1) \,\&\, F(2) \rightarrow F(y).$$

This formula is legitimate inasmuch as it becomes a true formula of the propositional calculus upon the replacement of the variable y by the numbers 1 and 2.

We can also assign a legitimate formula to axiom V.2 in an analogous manner.

Further, we can show that the rules for obtaining true formulae in the predicate calculus go over into rules for the corresponding formulae by virtue of which one obtains from legitimate formulae again legitimate formulae of the predicate calculus.

We consider, for instance, the first rule for binding by a quantifier. We assume that the formula

$$A \rightarrow B(x),$$

where A does not contain the variable x, is true and that the formula corresponding to it is a legitimate formula of the predicate calculus. This formula has the form:

$$A^{**} \rightarrow B^{**}(x), \tag{2}$$

where A^{**} and B^{**} are formulae corresponding to A and B. Since formula (2) is legitimate by assumption, the formulae

$$A^{**} \rightarrow B^{**}(1) \quad \text{and} \quad A^{**} \rightarrow B^{**}(2)$$

are also legitimate. But then the formula

$$A^{**} \rightarrow B^{**}(1) \,\&\, B^{**}(2)$$

is also legitimate, and this is the formula corresponding to

$$A \rightarrow (x)B(x).$$

Carrying out the proof for all rules of the predicate calculus, we may show that there corresponds a legitimate formula to each true formula of the predicate calculus.

We now consider the formula corresponding to the formula (1), under investigation. This, obviously, is

$$F(1) \vee F(2) \rightarrow F(1) \,\&\, F(2). \tag{3}$$

Since formula (1) does not contain free variables, formula (3)—if it is legitimate—must be a true formula in the propositional calculus. It is, however, easy to see that formula (3) is not true. In fact, for a predicate F for which $F(1)$ has the value T and $F(2)$ has the value F, formula (3) goes over into

$$T \vee F \rightarrow T \,\&\, F,$$

i.e. the proposition takes on the value F. It follows from this that formula (1) cannot be true in the predicate calculus, which is what we required to prove.

§7. Certain theorems of the predicate calculus

The assertion that formula A is true or deducible in the predicate calculus will be written as in the propositional calculus:

$$\vdash A.$$

Since all formulae which are true in the propositional calculus are also true in the predicate calculus, by performing substitutions in true formulae of the propositional calculus, we shall obtain true formulae in the predicate calculus.

EXAMPLES:

1. Replacing A by $F(x)$ in the true formula of the propositional calculus

$$A \vee \bar{A},$$

we obtain a true formula in the predicate calculus:

$$\vdash F(x) \vee \bar{F}(x).$$

2. Replacing A by $F(x)$ and B by $(y)G(y)$ in the true formula

$$A \rightarrow A \vee B,$$

we obtain

$$\vdash F(x) \rightarrow F(x) \vee (y)G(y).$$

3. Replacing B by $(\exists x)F(x)$ and C by $(y)H(y)$ in the true formula

$$A \rightarrow (B \,\&\, C \rightarrow B) \,\&\, A,$$

we obtain

$$\vdash A \rightarrow ((\exists x)F(x) \,\&\, (y)H(y) \rightarrow (\exists x)F(x)) \,\&\, A.$$

We note that discovering the truth of a formula in the propositional calculus presents no difficulty; it is not necessary to carry out its deduction in the system of Chapter II, for it is sufficient simply to show that the formula is identically true in the sense of propositional algebra.

One can obtain many true formulae of the predicate calculus by means of substitutions in true formulae of the propositional calculus; it is, however, impossible to deduce every true formula of the predicate calculus in this way.

All multiplication rules deduced for the propositional calculus remain valid for the predicate calculus also. The rule of compound substitution (see page 52) and likewise the rule for compound inference are formed by the successive application of the fundamental rule of substitution (or the fundamental rule of inference) and therefore they remain valid for the predicate calculus also.

We shall not give the derivations of all these rules because they are obtained by a repetition of the corresponding proofs in the propositional calculus. As an example, we shall prove only the validity of the syllogism rule:

$$\frac{\boldsymbol{A} \rightarrow \boldsymbol{B},\ \boldsymbol{B} \rightarrow \boldsymbol{C}}{\boldsymbol{A} \rightarrow \boldsymbol{C}}$$

(under the assumption that $\boldsymbol{A} \rightarrow \boldsymbol{C}$ is a formula).

In the propositional calculus we deduced this rule from the true formula

$$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)).$$

But since the substitution rule also holds in the predicate calculus,

$$\vdash (\boldsymbol{A} \rightarrow \boldsymbol{B}) \rightarrow ((\boldsymbol{B} \rightarrow \boldsymbol{C}) \rightarrow (\boldsymbol{A} \rightarrow \boldsymbol{C}))$$

is true whatever the formulae $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ of the predicate calculus are. Collision of variables cannot arise in the formation of this formula because otherwise we should have collision of variables in any of the formulae $\boldsymbol{A}$, $\boldsymbol{B}$ or $\boldsymbol{C}$, or among variables of any pair of these formulae. But since every pair of formulae $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ occurs in one of the formulae $\boldsymbol{A} \rightarrow \boldsymbol{B}$, $\boldsymbol{B} \rightarrow \boldsymbol{C}$ and $\boldsymbol{A} \rightarrow \boldsymbol{C}$, the collision of variables would occur in at least one of these formulae, which by hypothesis is not the case.

The formulae $\boldsymbol{A} \rightarrow \boldsymbol{B}$ and $\boldsymbol{B} \rightarrow \boldsymbol{C}$ are true by hypothesis. Applying the syllogism rule, we obtain that the formula $\boldsymbol{A} \rightarrow \boldsymbol{C}$ is also true.

The formulae

$$\vdash \boldsymbol{A} \rightarrow \boldsymbol{T}$$

and

$$\vdash \boldsymbol{F} \rightarrow \boldsymbol{A}$$

are true in the predicate calculus, where $\boldsymbol{T}$ denotes an arbitrary true formula and $\boldsymbol{F}$ denotes an arbitrary false formula.

We shall derive the following *product rule* for the predicate calculus.

If the formula $A(x)$, containing the free object variable x, is true, then the formula $(x)A(x)$ is also true in the predicate calculus.

Let us assume that

$$\vdash A(x)$$

holds.

In virtue of the fact that

$$\vdash A \rightarrow T$$

holds, where T is an arbitrary true formula, we have

$$\vdash A \rightarrow A(x).$$

Applying the first rule for binding by a quantifier, we obtain

$$\vdash A \rightarrow (x)A(x).$$

We can assume that x does not occur in the formula A (A can always be chosen this way). Replacing A in the last formula by an arbitrary formula, we have

$$\vdash T \rightarrow (x)A(x).$$

Applying the inference rule, we obtain

$$\vdash (x)A(x).$$

Thus, if $\vdash A(x)$ holds, then $\vdash (x)A(x)$ also holds, and we have proved the rule which can be written as follows:

$$\frac{(x)}{(x)A(x)}.$$

We shall call the rule just obtained the product rule for binding by a quantifier. It is obviously applicable to an arbitrary object variable.

Applying this rule, we can deduce further true formulae.

EXAMPLES:

1. Applying the rule

$$\frac{A(x)}{(x)A(x)}$$

to

$$\vdash F(x) \vee \overline{F}(x),$$

we obtain

$$\vdash (x)(F(x) \vee \overline{F}(x)).$$

2. Applying the product rule for binding by a quantifier, from the true formula

$$F(x) \rightarrow (G(y) \rightarrow F(x)),$$

which is the result of substitutions in axioms II.1, we obtain

$$\vdash (y)(F(x) \rightarrow (G(y) \rightarrow F(x))).$$

Applying this same rule to the last formula once more, we have

$$\vdash (x)(y)(F(x) \rightarrow (G(y) \rightarrow F(x))).$$

§8. Deduction theorem

We shall now prove for the predicate calculus a theorem which is analogous to the deduction theorem which we obtained for the propositional calculus. This theorem will allow us to obtain deducible formulae in the predicate calculus without performing all the operations of formal deduction, and so to a significant degree to shorten the labour of deducing true formulae. We shall retain the name "deduction theorem" in the predicate calculus for this theorem also.

We first introduce the following definition:

We shall say that the formula $\boldsymbol{B}$ is deducible from the formula $\boldsymbol{A}$ if the formula $\boldsymbol{B}$ is deducible from the totality of all true formulae of the predicate calculus and the formula $\boldsymbol{A}$ by means of application of all the rules of the predicate calculus, in which connection both rules for binding by a quantifier, the rules for substitution in place of predicate variables and in place of free object variables must be applied only to predicate variables or object variables which do not occur in the formula $\boldsymbol{A}$, and if $\boldsymbol{A} \rightarrow \boldsymbol{B}$ is a formula.

We stated a preliminary—not completely precise—definition of "the deducibility of the formula $\boldsymbol{B}$ from the formula $\boldsymbol{A}$". We shall now give a precise definition of this concept. It is built up from the following steps.

1. Every true formula $\boldsymbol{B}$ of the predicate calculus is deducible from $\boldsymbol{A}$ provided simply that the expression $\boldsymbol{A} \rightarrow \boldsymbol{B}$ does not contain collisions of variables.
2. The formula $\boldsymbol{A}$ is deducible from the formula $\boldsymbol{A}$.
3. If the formulae $\boldsymbol{B}_1$ and $\boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2$ are deducible from the formula $\boldsymbol{A}$, then the formula $\boldsymbol{B}_2$ is also deducible from the formula $\boldsymbol{A}$.
4. If the formula $\boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2(x)$ is deducible from $\boldsymbol{A}$, and if $\boldsymbol{B}_1$ and $\boldsymbol{A}$ do not contain the variable x, then the formula $\boldsymbol{B}_1 \rightarrow (x)\boldsymbol{B}_2(x)$ is also deducible from the formula $\boldsymbol{A}$.
5. If the formula $\boldsymbol{B}_2(x) \rightarrow \boldsymbol{B}_1$ is deducible from $\boldsymbol{A}$, and if x occurs neither in $\boldsymbol{B}_1$ nor in $\boldsymbol{A}$, then the formula $(\exists x)\boldsymbol{B}_2(x) \rightarrow \boldsymbol{B}_1$ is also deducible from $\boldsymbol{A}$.
6. If $\boldsymbol{B}$ is deducible from $\boldsymbol{A}$, then the formula $\boldsymbol{B}'$ obtained from $\boldsymbol{B}$ by an arbitrary renaming of the bound variables not leading to a collision of the variables with $\boldsymbol{A}$ is deducible from $\boldsymbol{A}$.
7. If $\boldsymbol{B}$ is deducible from $\boldsymbol{A}$, then the formula $\boldsymbol{B}'$ which is obtained from $\boldsymbol{B}$ by means of a substitution in a free object variable, which does not occur in $\boldsymbol{A}$, is also deducible from $\boldsymbol{A}$ if this substitution does not lead to a collision of the variables with $\boldsymbol{A}$.
8. If $\boldsymbol{B}$ is deducible from $\boldsymbol{A}$ and if the formula $\boldsymbol{B}'$ which is obtained from $\boldsymbol{B}$ by means of a substitution in a propositional variable or predicate variable, and if this propositional variable or predicate variable is not contained in the formula $\boldsymbol{A}$, and if, furthermore, this substitution does not lead to a collision of the variables with $\boldsymbol{A}$, then $\boldsymbol{B}'$ is also deducible from $\boldsymbol{A}$.

DEDUCTION THEOREM. *If the formula $\boldsymbol{B}$ is deducible from the formula $\boldsymbol{A}$, then the formula $\boldsymbol{A} \to \boldsymbol{B}$ is deducible in the predicate calculus.*

Here we of course assume that $\boldsymbol{A}$ and $\boldsymbol{B}$ are such that $\boldsymbol{A} \to \boldsymbol{B}$ is a formula, i.e. collision of variables does not arise between $\boldsymbol{A}$ and $\boldsymbol{B}$. Moreover, whatever the formula $\boldsymbol{B}$, one can rename its object variables so that the resultant formula $\boldsymbol{B}'$ does not lead to collision of variables with $\boldsymbol{A}$ and allows one to form the formula $\boldsymbol{A} \to \boldsymbol{B}'$. In this case, the deduction theorem can be formulated for the formulae $\boldsymbol{A}$ and $\boldsymbol{B}'$.

To prove the deduction theorem, it suffices to show that it holds in the following cases*:

(a) It holds for an arbitrary formula $\boldsymbol{B}$ which is true in the predicate calculus.

(b) It holds when $\boldsymbol{B}$ is the formula $\boldsymbol{A}$.

(c) If the theorem is valid for the formulae $\boldsymbol{B}_1$ and $\boldsymbol{B}_1 \to \boldsymbol{B}_2$, then it is also valid for the formula $\boldsymbol{B}_2$.

(d) If the theorem is valid for the formula $\boldsymbol{B}_1 \to \boldsymbol{B}_2(x)$, where x occurs neither in $\boldsymbol{B}$ nor in $\boldsymbol{A}$, then it is also valid for the formula $\boldsymbol{B}_1 \to (x)\boldsymbol{B}_2(x)$.

(e) If the theorem is valid for the formula $\boldsymbol{B}_2(x) \to \boldsymbol{B}_1$, where x occurs neither in $\boldsymbol{B}_1$ nor in $\boldsymbol{A}$, then it is also valid for the formula $(\exists x)\boldsymbol{B}_2(x) \to \boldsymbol{B}_1$.

(f) If the theorem is valid for $\boldsymbol{B}$, then it is also valid for an arbitrary formula $\boldsymbol{B}'$, which is obtained from $\boldsymbol{B}$ by a renaming of the bound variables, provided only that this renaming does not lead to a collision of the variables with the formula $\boldsymbol{A}$.

(g) If the theorem is valid for the formula $\boldsymbol{B}$, then it is also valid for $\boldsymbol{B}'$, which is obtained from $\boldsymbol{B}$ by a substitution in a free object variable not occurring in $\boldsymbol{A}$ provided only that this substitution does not lead to a collision of the variables with the formula $\boldsymbol{A}$.

(h) If the theorem is valid for the formula $\boldsymbol{B}$, then it is also valid for the formula $\boldsymbol{B}'$ obtained from $\boldsymbol{B}$ by a substitution in a propositional variable or predicate variable not contained in $\boldsymbol{A}$ under the condition that a collision of variables does not arise between $\boldsymbol{A}$ and $\boldsymbol{B}'$.

The validity of case (a) follows from the fact that every formula of the form $\boldsymbol{A} \to \boldsymbol{T}$, where $\boldsymbol{T}$ is a true formula, is also true. Consequently, if $\boldsymbol{B}$ is a true formula, then $\boldsymbol{A} \to \boldsymbol{B}$ is also a true formula.

The validity of case (b) is obvious.

We shall prove case (c). Let $\boldsymbol{B}_1$ and $\boldsymbol{B}_1 \to \boldsymbol{B}_2$ be formulae deducible from $\boldsymbol{A}$ for which the deduction theorem is valid.

We take axiom I.2:

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)).$$

* The proof is again by induction on the number of applications of a rule of deduction in the deduction of $\boldsymbol{B}$ from $\boldsymbol{A}$. If the theorem is valid for p applications of rules of deduction, then in a deduction which makes $p + 1$ applications of rules of deduction the last will be one of the rules listed in cases (c) to (h). [R. L. G.]

By means of substitutions in this axiom, we obtain the following true formula of the predicate calculus:

$$\vdash (\boldsymbol{A} \rightarrow (\boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2)) \rightarrow ((\boldsymbol{A} \rightarrow \boldsymbol{B}_1) \rightarrow (\boldsymbol{A} \rightarrow \boldsymbol{B}_2)).$$

Since we have

$$\vdash \boldsymbol{A} \rightarrow (\boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2) \text{ and } \vdash \boldsymbol{A} \rightarrow \boldsymbol{B}_1$$

by assumption, we obtain, applying the compound rule of inference, that

$$\vdash \boldsymbol{A} \rightarrow \boldsymbol{B}_2.$$

To prove (d), we assume that our theorem is valid for the formula $\boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2(x)$ which is deducible from $\boldsymbol{A}$ (where x occurs neither in $\boldsymbol{B}_1$ nor in $\boldsymbol{A}$). This means that

$$\vdash \boldsymbol{A} \rightarrow (\boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2(x))$$

holds. Applying the rule for the combination of antecedents,

$$\frac{\boldsymbol{A} \rightarrow (\boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2(x))}{\boldsymbol{A} \,\&\, \boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2(x)},$$

we obtain that

$$\vdash \boldsymbol{A} \,\&\, \boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2(x).$$

Applying the first rule for binding by a quantifier, we shall have

$$\vdash \boldsymbol{A} \,\&\, \boldsymbol{B}_1 \rightarrow (x)\boldsymbol{B}_2(x).$$

Now, applying the rule for the separation of antecedents,

$$\frac{\boldsymbol{A} \,\&\, \boldsymbol{B}_1 \rightarrow (x)\boldsymbol{B}_2(x)}{\boldsymbol{A} \rightarrow (\boldsymbol{B}_1 \rightarrow (x)\boldsymbol{B}_2(x))},$$

we obtain the required formula:

$$\vdash \boldsymbol{A} \rightarrow (\boldsymbol{B}_1 \rightarrow (x)\boldsymbol{B}_2(x)).$$

We shall now prove (e). The validity of this case is proved in the same way as was case (d); only in this case we need to utilize the rule for the commutation of antecedents:

$$\frac{\boldsymbol{A} \rightarrow (\boldsymbol{B} \rightarrow \boldsymbol{C})}{\boldsymbol{B} \rightarrow (\boldsymbol{A} \rightarrow \boldsymbol{C})}.$$

The validity of cases (f) and (g) is obvious.

Concerning case (h), we note that the validity of this case is also clear. In fact, $\vdash \boldsymbol{A} \rightarrow \boldsymbol{B}$ holds for a formula $\boldsymbol{B}$ which is deducible from $\boldsymbol{A}$. If $\boldsymbol{B}'$ is the result of substitution in a variable proposition or variable predicate which does not occur in $\boldsymbol{A}$, then $\boldsymbol{A} \rightarrow \boldsymbol{B}'$ is the result of the same substitution in the formula $\boldsymbol{A} \rightarrow \boldsymbol{B}$. Therefore, $\boldsymbol{A} \rightarrow \boldsymbol{B}'$ is also a true formula in the predicate calculus.

§9. Further theorems in the predicate calculus

THEOREM 1. $\vdash (x)F(x) \rightarrow (\exists x)F(x)$.

Proof. We take axioms V.1 and V.2:

$$(x)F(x) \rightarrow F(y) \quad \text{and} \quad F(y) \rightarrow (\exists x)F(x).$$

Applying the syllogism rule, we obtain the required formula.

We introduce the sign $\sim$, defining it in the same way as in the propositional calculus, i.e. we shall assume that the expression $\boldsymbol{A} \sim \boldsymbol{B}$ represents the formula

$$(\boldsymbol{A} \rightarrow \boldsymbol{B}) \,\&\, (\boldsymbol{B} \rightarrow \boldsymbol{A}).$$

We shall retain the nomenclature *equivalence sign* for the sign $\sim$, and we shall call formulae of the form $\boldsymbol{A} \sim \boldsymbol{B}$ *equivalences.*

THEOREM 2. $(x)(y)F(x, y) \sim (y)(x)F(x, y)$.

Proof. Applying axiom V.1 twice, we find that

$$(x)(y)F(x, y) \rightarrow F(u, v).$$

We apply to this formula the first rule for binding a variable, binding first the variable u and then the variable v. Then we obtain

$$(x)(y)F(x, y) \rightarrow (v)(u)F(u, v).$$

Performing in this formula a renaming of variables, replacing u by x and v by y, we obtain the formula

$$(x)(y)F(x, y) \rightarrow (y)(x)F(x, y).$$

The reverse implication is proved in the same way. Finally, applying the rule

$$\frac{\boldsymbol{A}, \boldsymbol{B}}{\boldsymbol{A} \,\&\, \boldsymbol{B}},$$

we obtain the required equivalence.

THEOREM 3. $(\exists x)(y)F(x, y) \rightarrow (y)(\exists x)F(x, y)$.

Proof. By substitution in axiom IV.1 and replacement of the free variables, we obtain

$$\vdash (y)F(x, y) \rightarrow F(x, v).$$

In the same way, from axiom V.2, we have

$$\vdash F(x, v) \rightarrow (\exists w)F(w, v).$$

Applying the syllogism rule to the formulae just obtained, we find that

$$\vdash (y)F(x, y) \rightarrow (\exists w)\, F(w, v).$$

To the last formula we apply first the second rule for binding by a quantifier and then the first; we obtain that

$$\vdash (\exists x)(y)F(x, y) \rightarrow (v)(\exists w)F(w, v).$$

Finally, applying the rule for renaming bound variables, we obtain the required formula.

The reverse implication is not deducible. For, if it were deducible, then we would have the universally valid law expressed by the formula

$$(x)(\exists y)F(x, y) \rightarrow (\exists y)(x)F(x, y).$$

It is easily seen, however, that such a law quickly leads to a contradiction. Let us apply it to the sequence of natural numbers.

Let $F(x, y)$ denote "The natural number x is less than the natural number y". In this case, it is asserted in the antecedent of the implication under consideration that for every natural number x there exists a larger natural number y. This assertion is perfectly valid for the sequence of natural numbers. In this case the consequent must also be valid, i.e.

$$(\exists y)(x)F(x, y).$$

However in our case this assertion expresses the following: "there exists a natural number y such that every natural number x is less than the number y". But this assertion is obviously invalid.

Our reasoning is not a rigorous proof of the non-deducibility of the formula

$$(x)(\exists y)F(x, y) \rightarrow (\exists y)(x)F(x, y)$$

in the predicate calculus. But it is not too difficult to give a rigorous proof that this formula is indeed non-deducible; we shall, however, not stop to do this.

THEOREM 4. $\vdash (x)[F(x) \rightarrow G(x)] \rightarrow ((x)F(x) \rightarrow (x)G(x)).$

Proof. To prove the truth of this formula, we shall employ the deduction theorem. We shall show that the formula

$$(x)F(x) \rightarrow (x)G(x)$$

is deducible from the formula

$$(x)(F(x) \rightarrow G(x)). \tag{1}$$

In fact, the formula

$$(x)(F(x) \rightarrow G(x)) \rightarrow (F(y) \rightarrow G(y))$$

is obtained by means of a substitution in axiom V.1 and it is therefore a true formula in the predicate calculus. Consequently, this formula is deducible from an arbitrary formula and, in particular, from formula (1). Applying the rule of inference, we find that the formula $F(y) \rightarrow G(y)$ is deducible from formula (1).

We write down the true formula of the propositional calculus:

$$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)).$$

By substitutions, we obtain from it

$$\vdash ((x)F(x) \rightarrow F(y)) \rightarrow ((F(y) \rightarrow G(y)) \rightarrow ((x)F(x) \rightarrow G(y))).$$

Both antecedents of this formula are deducible from formula (1) (the first one is axiom V.1). Applying the rule of inference twice, we find that the formula

$$(x)F(x) \rightarrow G(y)$$

is deducible from formula (1).

Finally, applying to the last formula the first rule for binding by a quantifier and then renaming the bound variable y, we find that the formula

$$(x)F(x) \rightarrow (x)G(x)$$

is deducible from formula (1). Applying the deduction theorem, we obtain the required formula.

It may appear strange that in one case of the proof of Theorem 4 we did not apply the syllogism rule and instead of it we used the formula

$$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)).$$

We did this because we have not proved the validity of the syllogism rule for the concept of deducibility which we introduced in the deduction theorem. However, this rule is also valid for deduction in the sense of the deduction theorem, and its proof can be carried out in the most general form in exactly the same way as it was actually done in Theorem 4 for the particular case. In the sequel, we shall also apply the syllogism rule for deduction in the sense of the deduction theorem.

THEOREM 5. $\vdash (x)(F(x) \rightarrow G(x)) \rightarrow ((\exists x)F(x) \rightarrow (\exists x)G(x))$.

Proof. We shall show that the right member of the implication is deducible from the left member. From the true formula

$$\vdash (x)(F(x) \rightarrow G(x)) \rightarrow (F(y) \rightarrow G(y)),$$

applying the rule of inference, we find that the formula $F(x) \rightarrow G(x)$ is deducible from the formula

$$(x)(F(x) \rightarrow G(x)). \tag{2}$$

From the formulae

$$G(y) \rightarrow (\exists x)G(x)$$

and

$$F(y) \rightarrow G(y),$$

which are deducible from formula (2) (the first because it is true), applying the syllogism rule, we obtain the formula

$$F(y) \rightarrow (\exists x)G(x),$$

which is also deducible from formula (2).

Applying the second rule for binding by a quantifier with respect to the variable y to formula (3), we obtain, after a renaming of bound variables, the formula

$$(\exists x)F(x) \rightarrow (\exists x)G(x),$$

which, consequently, is also deducible from formula (2). [We can apply the

rule for binding by a quantifier to formula (3) since the variable y does not occur in formula (2).]

As a result, on the basis of the deduction theorem, we can conclude that

$$\vdash (x)[F(x) \to G(x)] \to [(\exists x)F(x) \to (\exists x)G(x)]$$

holds, and the theorem is proved.

REMARKS ON THEOREM 5. It is easily seen that the following formula is true:

$$\vdash (x)(F(x) \sim G(x)) \to ((\exists x)F(x) \sim (\exists x)G(x)).$$

In fact, we showed in Theorem 5 that the formula

$$(\exists x)F(x) \to (\exists x)G(x)$$

is deducible from the formula

$$F(y) \to G(y).$$

Clearly, the formula

$$(\exists x)G(x) \to (\exists x)G(x)$$

is deducible from the formula $G(y) \to F(y)$. But both the formulae

$$F(y) \to G(y) \quad \text{and} \quad G(y) \to F(y)$$

are deducible from the formula

$$(x)(F(x) \sim G(x)),$$

as can easily be seen. In this case, the formulae

$$(\exists x)G(x) \to (\exists x)F(x) \quad \text{and} \quad (\exists x)F(x) \to (\exists x)G(x)$$

are also deducible from this formula.

By means of substitutions in the formula

$$\vdash A \to (B \to A \,\&\, B),$$

which is true in the propositional calculus, we obtain

$$\vdash ((\exists x)F(x) \to (\exists x)G(x)) \to (((\exists x)G(x) \to (\exists x)F(x)) \\ \to ((\exists x)F(x) \sim (\exists x)G(x))).$$

Applying the rule of inference twice, we see that the formula

$$(\exists x)F(x) \sim (\exists x)G(x)$$

is deducible from the formula $(x)(F(x) \sim G(x))$ from which it follows, by virtue of the deduction theorem, that

$$\vdash (x)(F(x) \sim G(x)) \to ((\exists x)F(x) \sim (\exists x)G(x)).$$

THEOREM 6. $\vdash (x)(F(x) \sim G(x)) \to ((x)F(x) \sim (x)G(x))$.

Proof. We shall show that the consequent is deducible from the antecedent. The formula $F(y) \sim G(y)$ is deducible from the formula

$$(x)(F(x) \sim G(x)). \tag{4}$$

The formula $F(y) \sim G(y)$ can be written in the form

$$(F(y) \rightarrow G(y)) \mathbin{\&} (G(y) \rightarrow F(y)).$$

We consider the two true formulae:

$$(F(y) \rightarrow G(y)) \mathbin{\&} (G(y) \rightarrow F(y)) \rightarrow (F(y) \rightarrow G(y)),$$
$$(F(y) \rightarrow G(y)) \mathbin{\&} (G(y) \rightarrow F(y)) \rightarrow (G(y) \rightarrow F(y)).$$

Applying the rule of inference, it is easily shown that both the formulae $F(y) \rightarrow G(y)$ and $G(y) \rightarrow F(y)$ are deducible from formula (4). From the true formula $(x)F(x) \rightarrow F(y)$ and the formula $F(y) \rightarrow G(y)$ [which is deducible from the antecedent (4)], applying the syllogism rule, we show that the formula $(x)F(x) \rightarrow G(y)$ is deducible from formula (4). Applying the first rule for binding by a quantifier and renaming variables, we find that the formula

$$(x)F(x) \rightarrow (x)G(x)$$

is deducible from formula (4). It is proved in the same way that the reverse implication

$$(x)G(x) \rightarrow (x)F(x)$$

is deducible from formula (4).

We now consider the following true formula of the propositional calculus:

$$A \rightarrow (B \rightarrow A \mathbin{\&} B).$$

By means of a compound substitution, we obtain from this the true formula:

$$((x)F(x) \rightarrow (x)G(x)) \rightarrow [((x)G(x) \rightarrow (x)F(x)) \\ \rightarrow ((x)F(x) \rightarrow (x)G(x)) \mathbin{\&} ((x)G(x) \rightarrow (x)F(x))].$$

Both antecedents of this formula are deducible from formula (4). Applying the rule of inference twice, we find that the formula

$$((x)F(x) \rightarrow (x)G(x)) \mathbin{\&} ((x)G(x) \rightarrow (x)F(x)),$$

or, equivalently, that the formula

$$(x)F(x) \sim (x)G(x)$$

is deducible from formula (4). We prove Theorem 6 from this by applying the deduction theorem.

THEOREM 7.

(a) $(\exists x)F(x) \sim \overline{(x)\overline{F}(x)}$; (b) $(\exists x)\overline{F}(x) \sim \overline{(x)F(x)}$;

(c) $\overline{(\exists x)F(x)} \sim (x)\overline{F}(x)$; (d) $\overline{(\exists x)\overline{F}(x)} \sim (x)F(x)$.

Proof. Firstly the proof of 7a. By means of a substitution in axiom V.1, we obtain

$$\vdash (x)F(x) \rightarrow F(y).$$

Reversing the implication, using the rule

$$\frac{A \to B}{\bar{B} \to \bar{A}},$$

we have

$$\vdash \bar{F}(y) \to \overline{(x)F(x)}.$$

From the last formula and the true formula

$$F(y) \to \bar{\bar{F}}(y),$$

applying the syllogism rule, we obtain

$$\vdash F(y) \to \overline{(x)\bar{F}(x)}.$$

Applying the second rule for binding by a quantifier and renaming the bound variables, we have

$$\vdash (\exists x)F(x) \to \overline{(x)\bar{F}(x)}. \tag{5}$$

We shall now deduce the reverse implication. Applying the rule for reversing an implication to axiom V.2, we obtain

$$\vdash \overline{(\exists x)F(x)} \to \bar{F}(y).$$

Applying the first rule for binding by a quantifier and renaming the bound variables, we have

$$\vdash \overline{(\exists x)F(x)} \to (x)\bar{F}(x).$$

Reversing the implication, we obtain

$$\overline{(x)\bar{F}(x)} \to \overline{\overline{(\exists x)F(x)}}.$$

Applying the syllogism rule to the last formula and to the true formula

$$\overline{\overline{(\exists x)F(x)}} \to (\exists x)F(x),$$

we have

$$\overline{(x)\bar{F}(x)} \to (\exists x)F(x). \tag{6}$$

We now obtain 7a by applying the rule

$$\frac{A, B}{A \,\&\, B}$$

to formulae (5) and (6).

We now prove 7b. Consider the true formula:

$$\vdash F(x) \sim \bar{\bar{F}}(x).$$

Applying the derived rule for binding by a quantifier, we have

$$\vdash (x)(F(x) \sim \bar{\bar{F}}(x)).$$

We make a substitution in the true formula proved in Theorem 6, replacing $G(x)$ by $\bar{\bar{F}}(x)$; we obtain

$$\vdash (x)(F(x) \sim \bar{\bar{F}}(x)) \to ((x)F(x) \sim (x)\bar{\bar{F}}(x)).$$

Applying the rule of inference to the last two formulae, we have

$$\vdash (x)F(x) \sim (x)\bar{\bar{F}}(x).$$

We consider both implications comprised in this formula:

$$(x)F(x) \rightarrow (x)\bar{\bar{F}}(x) \quad \text{and} \quad (x)\bar{\bar{F}}(x) \rightarrow (x)F(x).$$

We reverse both these implications and then combine the reversals thus obtained in the formula

$$\vdash \overline{(x)F(x)} \sim \overline{(x)\bar{\bar{F}}(x)}.$$

We perform a substitution in formula 7a; replacing $F(x)$ by $\bar{F}(x)$, we obtain

$$\vdash (\exists x)\bar{F}(x) \sim \overline{(x)\bar{\bar{F}}(x)}.$$

We now obtain formula 7b by applying the rule

$$\frac{A \sim B,\, B \sim C}{A \sim C}$$

to the last two formulae.

The truth of formulae 7c and 7d is easily proved, by means of the rule for the reversal of implication, from formulae 7a and 7b.

THEOREM 8. $\vdash (A \rightarrow (x)F(x)) \sim (x)(A \rightarrow F(x))$.

Proof. We shall prove the first implication:

$$(A \rightarrow (x)F(x)) \rightarrow (x)(A \rightarrow F(x)).$$

We shall first prove that $F(y)$ is deducible from the formula

$$(A \rightarrow (x)F(x)) \,\&\, A. \tag{7}$$

In fact, the formulae $A \rightarrow (x)F(x)$ and A are obviously deducible from formula (7). Applying the rule of inference to these formulae, we see that the formula $(x)F(x)$ is deducible from formula (7). Axiom V.1, $(x)F(x) \rightarrow F(y)$, being a true formula, is deducible from (7). Applying the rule of inference to the formulae $(x)F(x)$ and $(x)F(x) \rightarrow F(y)$, we find that $F(y)$ is deducible from (7).

We conclude, on the basis of the deduction theorem, that

$$\vdash (A \rightarrow (x)F(x)) \,\&\, A \rightarrow F(y).$$

Applying the rule

$$\frac{A \,\&\, B \rightarrow C}{A \rightarrow (B \rightarrow C)},$$

we obtain

$$\vdash (A \rightarrow (x)F(x)) \rightarrow (A \rightarrow F(y)),$$

from which we have, applying the rule for binding by a quantifier and then renaming the bound variables, that

$$\vdash (A \rightarrow (x)F(x)) \rightarrow (x)(A \rightarrow F(x)). \tag{8}$$

We shall now prove the reverse implication:

$$\vdash (x)(A \rightarrow F(x)) \rightarrow (A \rightarrow (x)F(x)).$$

We shall show that the consequent is deducible from the antecedent. In fact, from the formula

$$(x)(A \to F(x)) \tag{9}$$

and the true formula

$$\vdash (x)(A \to F(x)) \to (A \to F(y)),$$

applying the rule of inference, we find that the formula $A \to F(y)$ is deducible from formula (9). Applying the first rule for binding by a quantifier and renaming the bound variables, we find that the formula

$$A \to (x)F(x)$$

is deducible from formula (9).

On the basis of the deduction theorem, we conclude that

$$\vdash (x)(A \to F(x)) \to (A \to (x)F(x)).$$

The truth of the equivalence

$$\vdash (x)(A \to F(x)) \sim (A \to (x)F(x))$$

now follows from the truth of the implications just proved.

§10. Equivalent formulae

As in the propositional calculus, we shall say here that the formulae A and B are *equivalent* if

$$\vdash A \sim B$$

holds.

Since the rules

$$\frac{A \sim B, B \sim C}{A \sim C} \quad \text{and} \quad \frac{A \sim B}{B \sim A}$$

are also valid in the predicate calculus, the equivalence relation is transitive and symmetric. We shall show that *if in a formula A of the predicate calculus we replace an arbitrary part by an equivalent formula and if the expression A' obtained as the result of this replacement is also a formula and contains all the free object variables of the formula A, then A and A' are equivalent.*

We shall not carry out the proof of this assertion completely since it is to a significant degree a repetition of the proof of the analogous proposition in the propositional calculus (see Chapter II, §6). The proof is by induction on the operations of forming formulae.

We note first that our assertion holds for elementary formulae, i.e. for propositional variables and predicate variables. This is indeed obvious since every elementary formula has only one part—namely itself.

We shall assume that our assertion is valid for the formulae A and B. Then it is also valid for the formulae $A \& B$, $A \vee B$, $A \to B$ and $\bar{A}$. The proof of this is a verbatim repetition of the line of reasoning followed in the propositional calculus.

We shall prove that if our assertion is valid for the formula $A(x)$, then it is also valid for the formulae

$$(x)A(x) \quad \text{and} \quad (\exists x)A(x).$$

The proof will be given in detail for the formula $(x)A(x)$.

The part of this formula which is being replaced is, by definition, either all or a part of the formula $A(x)$. In the first case, our assertion is obvious. In the second case, by virtue of the induction hypothesis, the formula $A(x)$ goes over into the formula $A'(x)$ which is equivalent to the formula $A(x)$. [The variable x must be retained in the formula $A'(x)$ because otherwise the expression $(x)A'(x)$ would not be a formula.] Thus, $A(x)$ and $A'(x)$ are equivalent; this means that the formula $A(x) \sim A'(x)$ is true and, consequently, the formulae

$$A(x) \to A'(x) \quad \text{and} \quad A'(x) \to A(x)$$

are also true. Performing a substitution in the formula of Theorem 4, §9, we obtain

$$\vdash (x)(A(x) \to A'(x)) \to ((x)A(x) \to (x)A'(x)).$$

Moreover, applying the derived rule for binding by a quantifier to the formula $A(x) \to A'(x)$, we have

$$\vdash (x)(A(x) \to A'(x)).$$

Applying the rule of inference to the last two formulae, we obtain

$$\vdash (x)A(x) \to (x)A'(x).$$

The truth of the reverse implication is proved in an analogous manner with the aid of the formula $A'(x) \to A(x)$. Since both implications are true, we have

$$\vdash (x)A(x) \sim (x)A'(x)$$

and, consequently, our assertion is proved for the formula $(x)A(x)$.

We shall now prove the theorem for the formula $(\exists x)A(x)$.

Reversing the implication $A(x) \to A'(x)$, we obtain

$$\vdash \bar{A}'(x) \to \bar{A}(x).$$

Starting with this formula, we deduce, just as in the preceding case, that

$$\vdash (x)\bar{A}'(x) \to (x)\bar{A}(x).$$

Reversing this implication, we have

$$\vdash \overline{(x)\bar{A}(x)} \to \overline{(x)\bar{A}'(x)}. \tag{1}$$

By means of a substitution in 7a, we have that

$$\vdash (\exists x)A(x) \sim \overline{(x)\bar{A}(x)}$$

and

$$\vdash (\exists x)A'(x) \sim \overline{(x)\bar{A}'(x)}.$$

We write out some implications comprised in these equivalences:

$$\vdash (\exists x)A(x) \rightarrow \overline{(x)\bar{A}(x)},$$

$$\vdash \overline{(x)\bar{A}'(x)} \rightarrow (\exists x)A'(x).$$

From these implications and formula (1), we obtain, with the aid of the syllogism rule,

$$\vdash (\exists x)A(x) \rightarrow (\exists x)A'(x).$$

The reverse implication is proved in an analogous manner. Combining these implications, we have

$$\vdash (\exists x)A(x) \sim (\exists x)A'(x).$$

Our assertion is thus proved for all formulae.

The equivalence of the formulae $A \rightarrow B$ and $\bar{A} \vee B$, i.e. the truth of the assertion

$$\vdash (A \rightarrow B) \sim \bar{A} \vee B, \tag{2}$$

which is valid for the propositional calculus, holds also for the predicate calculus. The proof of this fact in the predicate calculus can be obtained by means of substitutions in formula (2) of the propositional calculus. Since replacement of arbitrary parts of a formula by equivalents reduces a given formula to an equivalent one, we can delete the sign $\rightarrow$ from a formula by replacing every part of the form $A \rightarrow B$ by the formula $\bar{A} \vee B$. After such a replacement, we obtain a formula which is equivalent to the given one.

Furthermore, we can find for every formula which does not contain the sign $\rightarrow$ a formula equivalent to it in which the negation signs apply only to elementary parts.

In fact, if any formula has the form $\overline{(x)A(x)}$ $[\overline{(\exists x)A(x)}]$, then the formula $(\exists x)\bar{A}(x)$ $[(x)\bar{A}(x)]$ is equivalent to it. Therefore, we can always bring the negation sign, appearing over a quantifier, under the sign of the quantifier, interchanging the universal quantifier with the existential quantifier, and conversely. The equivalences

$$\vdash \overline{A \& B} \sim \bar{A} \vee \bar{B},$$

$$\vdash \overline{A \vee B} \sim \bar{A} \& \bar{B},$$

$$\vdash \bar{\bar{A}} \sim A,$$

which we proved for the propositional calculus, hold also for the predicate calculus [their proofs in the predicate calculus are the same as in the propositional calculus (see Chapter II)]. It follows from this that a negation sign appearing over a sum can be brought inside—the sum going over into a product; a negation sign over a product can also be brought inside the formula, the product going over into a sum. But if a negation sign appears over a negation sign, then both these signs may be deleted.

By virtue of what we have stated, we can successively bring the negation sign inside a formula, replacing the formula by an equivalent formula.

Clearly, as a result of such operations, we arrive at a formula in which the negation sign applies only to its elementary parts.

Formulae not containing the sign $\rightarrow$ and those in which the sign — applies only to elementary parts will be called *reduced formulae*. It follows from the above discussion that for every formula A there exists a reduced formula which is equivalent to it. We shall call this formula the *reduced form of formula A*. We now consider an example of bringing a formula to the reduced form.

1. Consider $\overline{(\exists x)(A(x) \rightarrow B(x))}$. We first eliminate the sign $\rightarrow$. Replacing $A(x) \rightarrow B(x)$ by $\bar{A}(x) \vee B(x)$, we have

$$\overline{(\exists x)(\bar{A}(x) \vee B(x))}.$$

We then bring the exterior negation sign under the quantifier:

$$(x)\overline{(\bar{A}(x) \vee B(x))}.$$

Further, we bring the negation sign inside the sum:

$$(x)(\bar{\bar{A}}(x) \mathbin{\&} \bar{B}(x)).$$

We now delete the double negation sign:

$$(x)(A(x) \mathbin{\&} \bar{B}(x)).$$

The formula just obtained is equivalent to the given formula and is a reduced formula. Consequently, it is the reduced form of the initial formula.

The following equivalences which express the associativity and commutativity of sums and products and the two distributive laws which we proved in the propositional calculus are also valid for the predicate calculus (see Chapter II):

(a) $(A \vee B) \vee C \sim A \vee (B \vee C)$;

(b) $A \vee B \sim B \vee A$;

(c) $(A \mathbin{\&} B) \mathbin{\&} C \sim A \mathbin{\&} (B \mathbin{\&} C)$;

(d) $A \mathbin{\&} B \sim B \mathbin{\&} A$;

(e) $A \mathbin{\&} (B \vee C) \sim A \mathbin{\&} B \vee A \mathbin{\&} C$;

(f) $A \vee B \mathbin{\&} C \sim (A \vee B) \mathbin{\&} (A \vee C)$.

Therefore, in the predicate calculus, in situations in which equivalent formulae are interchangeable we shall also at times omit brackets in expressions in which the members are connected only by the sign & or only by the sign $\vee$. For example, we shall write the expression

$$(A \mathbin{\&} B) \mathbin{\&} C$$

in the form

$$A \mathbin{\&} B \mathbin{\&} C,$$

and the expression

$$A \vee (B \vee C) \vee (G \vee H)$$

in the form

$$A \vee B \vee C \vee G \vee H.$$

Of course, this expression is not a formula. We shall understand by such an expression the formula which can be obtained from it by arranging brackets in the proper way.

§11. The duality law

We shall now establish the concept of "dual formulae" for formulae which do not contain the symbol $\rightarrow$. We shall call the signs & and V the duals of one another. We shall also call the quantifiers (x) and $(\exists x)$ *duals*. We shall say that *the formula $\boldsymbol{B}$ is dual to the formula A if it can be obtained from the formula A by changing each of the symbols* &, V, (x), $(\exists x)$ *to its dual.*

It follows from the definition that the concept of duality is symmetric, i.e. if $\boldsymbol{B}$ is dual to $\boldsymbol{A}$, then $\boldsymbol{A}$ is also dual to $\boldsymbol{B}$.

The following are examples of dual formulae.

1. Given the formula

$$(x)(A \vee B(x) \,\&\, (B(y) \vee \bar{B}(x))),$$

we see that its dual is

$$(\exists x)(A \,\&\, B(x) \vee (B(y) \,\&\, \bar{B}(x))).$$

2. The dual of the formula

$$(x)(\exists y)(A(x, y) \vee ((z)A(x, z) \,\&\, (\exists z)\bar{A}(y, z)))$$

is

$$(\exists x)(y)(A(x, y) \,\&\, ((\exists z)A(x, z) \vee (z)\bar{A}(y, z))).$$

We shall furthermore give the concept "dual formula" an inductive definition which we shall find more convenient in the sequel in giving proofs.

(a) The dual of an elementary formula is this formula itself.

(b) If $\boldsymbol{A}^*$ is dual to $\boldsymbol{A}$ and $\boldsymbol{B}^*$ is dual to $\boldsymbol{B}$, then the duals to the formulae $\boldsymbol{A} \,\&\, \boldsymbol{B}$, $\boldsymbol{A} \vee \boldsymbol{B}$ are the formulae $\boldsymbol{A}^* \vee \boldsymbol{B}^*$, $\boldsymbol{A}^* \,\&\, \boldsymbol{B}^*$, respectively.

(c) If $\boldsymbol{A}^*$ is dual to $\boldsymbol{A}$, then $\bar{\boldsymbol{A}}^*$ is the dual formula to $\bar{\boldsymbol{A}}$.

(d) If $\boldsymbol{A}^*(x)$ is dual to the formula $\boldsymbol{A}(x)$, then the dual formula of $(x)\boldsymbol{A}(x)$ $[(\exists x)\boldsymbol{A}(x)]$ will be $(\exists x)\boldsymbol{A}^*(x)$ $[(x)\boldsymbol{A}^*(x)]$. Since the formulae under consideration do not contain the sign $\rightarrow$, by assumption, the concept of "dual formula" is completely defined.

The symmetry of the duality relation is a consequence of the given definition and is easily proved by induction. We shall denote the formula dual to $\boldsymbol{A}$ by $\boldsymbol{A}^*$.

LEMMA. *Let $A(A_1, \ldots, A_n, F_1, \ldots, F_m)$ be a formula in the predicate calculus, not containing the sign $\rightarrow$; let $A_1, \ldots, A_n$ be all the elementary propositions occurring in A and $F_1, \ldots, F_m$ all the elementary predicates occurring in $\boldsymbol{A}$. Then the following holds:*

$$\vdash \bar{\boldsymbol{A}}(A_1, \ldots, A_n, F_1, \ldots, F_m) \sim \boldsymbol{A}^*(\bar{A}_1, \ldots, \bar{A}_n, \bar{F}_1, \ldots, \bar{F}_m).$$

Proof. We shall prove this lemma by induction on the inductive definition of a dual formula.

For an elementary formula, the validity of the lemma is obvious inasmuch as an elementary formula is either a proposition or a predicate and its dual coincides with it.

Suppose the lemma is true for the formulae

$$B(B_1, \ldots, B_p, G_1, \ldots, G_q) \text{ and } A(A_1, \ldots, A_n, F_1, \ldots, F_m).$$

We then have that

$$\vdash \bar{A}(A_1, \ldots, A_n, F_1, \ldots, F_m) \sim A^*(\bar{A}_1, \ldots, \bar{A}_n, F_1, \ldots, F_m),$$
$$\vdash \bar{B}(B_1, \ldots, B_p, G_1, \ldots, G_q) \sim B^*(\bar{B}_1, \ldots, \bar{B}_p, \bar{G}_1, \ldots, \bar{G}_q).$$

We shall show that the lemma is then also true for the product and sum of the formulae A and B. In fact,

$$\vdash \overline{A \,\&\, B} \sim \bar{A} \vee \bar{B}.$$

Replacing $\bar{A}$ and $\bar{B}$, on the basis of the equivalences written down above, we obtain

$$\vdash \overline{A \,\&\, B} \sim A^*(\bar{A}_1, \ldots, \bar{A}_n, F_1, \ldots F_m) \vee B^*(\bar{B}_1, \ldots, \bar{B}_p, \bar{G}_1, \ldots, \bar{G}_q);$$

but, by definition,

$$A^*(\bar{A}_1, \ldots, \bar{A}_n, F_1, \ldots, F_m) \vee B^*(\bar{B}_1, \ldots, \bar{B}_p, \bar{G}_1, \ldots, \bar{G}_q)$$

represents

$$(A(\bar{A}_1, \ldots, \bar{A}_n, F_1, \ldots, F_m) \,\&\, B(\bar{B}_1, \ldots, \bar{B}_p, \bar{G}_1, \ldots, \bar{G}_q))^*,$$

and therefore

$$\vdash \overline{A \,\&\, B} \sim (A(\bar{A}_1, \ldots, \bar{A}_n, F_1, \ldots, F_m) \,\& \\ \& \, B(\bar{B}_1, \ldots, \bar{B}_p, \bar{G}_1, \ldots, \bar{G}_q))^*.$$

The validity of the lemma for $A \vee B$ is proved analogously.

We shall assume that the lemma is true for $A(A_1, \ldots, A_n, F_1, \ldots, F_m)$. We shall show that it is also true for $\bar{A}$. By virtue of our assumption, we shall have

$$\vdash \bar{A}(A_1, \ldots, A_n, F_1, \ldots, F_m) \sim A^*(\bar{A}_1, \ldots, \bar{A}_n, F_1, \ldots, F_m).$$

But if any two formulae are equivalent, then their negations are also equivalent. Therefore

$$\vdash \bar{\bar{A}}(A_1, \ldots, A_n, F_1, \ldots, F_m) \sim \bar{A}^*(\bar{A}_1, \ldots, \bar{A}_n, F_1, \ldots, F_m).$$

But, by definition, $\bar{A}^*$ is $(\bar{A})^*$. Consequently,

$$\vdash \bar{\bar{A}}(A_1, \ldots, A_n, F_1, \ldots, F_m) \sim (\bar{A}(\bar{A}_1, \ldots, \bar{A}_n, F_1, \ldots, F_m))^*,$$

and we have thus obtained the required equivalence.

Suppose the lemma is true for the formula $A(x, A_1, \ldots, F_m)$. We shall show that it is then true for the formulae

$$(x)A(x, A_1, \ldots, F_m) \quad \text{and} \quad (\exists x)A(x, A_1, \ldots, F_m).$$

On the basis of the induction hypothesis, we have

$$\vdash \bar{A}(x, A_1, \ldots, F_m) \sim A^*(x, \bar{A}_1, \ldots, \bar{F}_m);$$

but then

$$\vdash (\exists x)\bar{A}(x, A_1, \ldots, F_m) \sim (\exists x)A^*(x, \bar{A}_1, \ldots, \bar{F}_m).$$

By Theorem 7, §9,

$$\vdash (\exists x)\bar{A}(x, A_1, \ldots, F_m) \sim \overline{(x)A(x, A_1, \ldots, F_m)}.$$

From the last two equivalences we deduce that

$$\vdash \overline{(x)A(x, A_1, \ldots, F_m)} \sim (\exists x)A^*(x, \bar{A}_1, \ldots, \bar{F}_m).$$

By virtue of the definition, the right member of this formula represents

$$((x)A(x, \bar{A}_1, \ldots, \bar{F}_m))^*,$$

from which it follows that

$$\vdash \overline{(x)A(x, A_1, \ldots, F_m)} \sim ((x)A(x, \bar{A}_1, \ldots, \bar{F}_m))^*.$$

The lemma is proved for the formula $(\exists x)A(x, A_1, \ldots, F_m)$ in an analogous manner. We have thus proved the validity of the lemma for all formulae which do not contain the sign $\rightarrow$.

THEOREM. *If the formulae A and B are equivalent, then their duals are also equivalent.*

Proof. Let $A(A_1, \ldots, A_n, F_1, \ldots, F_m)$ and $B(B_1, \ldots, B_p, G_1, \ldots, G_q)$ be equivalent formulae where $A_1, \ldots, A_n, B_1, \ldots, B_p$ are all the propositional variables occurring in them and $F_1, \ldots, F_m, G_1, \ldots, G_q$ are all the predicate variables. As before, we shall mark dual formulae with an asterisk.

If the formulae A and B are equivalent, then their negations are also equivalent. We therefore have that

$$\vdash \bar{A}(A_1, \ldots, F_m) \sim \bar{B}(B_1, \ldots, G_q).$$

On the basis of the preceding lemma, the formula $\bar{A}(A_1, \ldots, F_m)$ is equivalent to the formula $A^*(\bar{A}_1, \ldots, \bar{F}_m)$ and the formula $\bar{B}(B_1, \ldots, G_q)$ is equivalent to the formula $B^*(\bar{B}_1, \ldots, \bar{G}_q)$. Replacing both parts of the true formula just obtained by equivalent formulae, we obtain a formula which is also true:

$$\vdash A^*(\bar{A}_1, \ldots, \bar{F}_m) \sim A^*(\bar{B}_1, \ldots, \bar{G}_q).$$

We perform a substitution in this formula, replacing A_i, B_i, F_i, G_i by $\bar{A}_i, \bar{B}_i, \bar{F}_i, \bar{G}_i$, respectively. We then obtain

$$\vdash A^*(\bar{\bar{A}}_1, \ldots, \bar{\bar{F}}_m) \sim B^*(\bar{\bar{B}}_1, \ldots, \bar{\bar{G}}_q).$$

Replacing in this formula every part of the form $\bar{\bar{A}}_i$ by A_i which is equivalent to it, $\bar{\bar{B}}_i$ by B_i, $\bar{\bar{F}}_i$ by F_i and $\bar{\bar{G}}_i$ by G_i, we obtain

$$\vdash A^*(A_1, \ldots, F_m) \sim B^*(B_1, \ldots, G_q).$$

The theorem just proved is called the *duality law*. It allows us to obtain from

equivalences, whose truth is established, other true equivalences and, like the deduction theorem, it thus facilitates the proof of the truth of certain formulae. For example, we proved (cf. Theorem 2, §9) that

$$\vdash (x)(y)F(x, y) \sim (y)(x)F(x, y).$$

By virtue of the duality law, we can assert the truth of the following formula:

$$\vdash (\exists x)(\exists y)F(x, y) \sim (\exists y)(\exists x)F(x, y).$$

From these equivalences one can deduce the following rule.

If quantifiers of the same type which occur adjacent to one another in a formula are interchanged, then the formula transforms into an equivalent one.

§12. Normal forms

We have already considered normal formulae and normal forms in Chapter III during the informal description of the logic of predicates. We shall introduce these same concepts for the predicate calculus.

We shall call a reduced formula *normal* if in the sequence of symbols forming the formula the quantifiers precede all the remaining symbols.

It can be proved that for every formula there exists a normal formula which is equivalent to it.

For the proof of this assertion, it is necessary to establish the validity of eertain transformations of equivalence which are analogous to the transformations of equivalance which we utilized in the informal logic of predicates for the same goal (see Chapter III, §2).

THEOREM 1. $\vdash (x)(A \vee F(x)) \sim A \vee (x)F(x)$.

Proof. It was proved in Theorem 8, §9, that

$$\vdash (x)(A \rightarrow F(x)) \sim A \rightarrow (x)F(x).$$

On the basis of the equivalence $A \rightarrow B \sim \bar{A} \vee B$, we replace both parts of the formula under consideration; we obtain

$$\vdash (x)(\bar{A} \vee F(x)) \sim \bar{A} \vee (x)F(x).$$

Substituting $\bar{A}$ in place of A in this formula, we have

$$\vdash (x)(\bar{\bar{A}} \vee F(x)) \sim \bar{\bar{A}} \vee (x)F(x).$$

Replacing $\bar{\bar{A}}$ by the elementary formula equivalent to it, we obtain the required formula.

THEOREM 2. $\vdash (x)(A \mathbin{\&} F(x)) \sim A \mathbin{\&} (x)F(x)$.

Proof. To prove this we shall apply the deduction theorem. We consider the formula

$$(x)(A \mathbin{\&} F(x)) \rightarrow A \mathbin{\&} (x)F(x).$$

We shall show that the consequent is deducible from the antecedent. In fact,

by the method which we have used many times, we prove the deducibility of the formulae $(x)F(x)$ and A from the antecedent. The formula

$$(x)F(x) \rightarrow (A \rightarrow A \,\&\, (x)F(x))$$

is a true formula of the predicate calculus. It is therefore deducible from the formula $(x)(A \,\&\, F(x))$. Applying the rule of inference twice to the two formulae $(x)F(x)$ and $(x)F(x) \rightarrow (A \rightarrow A \,\&\, (x)F(x))$, we see that the formula $A \,\&\, (x)F(x)$ is also deducible from $(x)(A \,\&\, F(x))$. Then, by virtue of the deduction theorem, we obtain

$$\vdash (x)(A \,\&\, F(x)) \rightarrow A \,\&\, (x)F(x). \tag{1}$$

We shall prove the validity of the reverse implication. To this end, we shall prove that the formula $(x)(A \,\&\, F(x))$ is deducible from the formula $A \,\&\, (x)F(x)$. In fact, as we have seen, the formulae A and $F(y)$ are deducible from the formula $A \,\&\, (x)F(x)$. In virtue of this, the formula $A \,\&\, F(y)$ is also deducible. From the true formula

$$A \,\&\, F(y) \rightarrow (A \rightarrow A \,\&\, F(y)),$$

applying the rule of inference, we obtain that the formula $A \rightarrow A \,\&\, F(y)$ is deducible from the formula $A \,\&\, (x)F(x)$. Applying the rule for binding by a quantifier and renaming the bound variables, we conclude that the formula

$$A \rightarrow (x)(A \,\&\, F(x))$$

is deducible from the formula $A \,\&\, (x)F(x)$. Eliminating, on the basis of the rule of inference, the antecedent A, we find that the formula $(x)(A \,\&\, F(x))$ is deducible from the formula $A \,\&\, (x)F(x)$. Then, on the basis of the deduction theorem, we conclude that

$$\vdash A \,\&\, (x)F(x) \rightarrow (x)(A \,\&\, F(x)). \tag{2}$$

The validity of the equivalence

$$\vdash A \,\&\, (x)F(x) \sim (x)(A \,\&\, F(x))$$

follows from the validity of implications (1) and (2).

On the basis of the duality law, the following theorems follow from Theorems 1 and 2:

THEOREM 1′. $\vdash (\exists x)(A \,\&\, F(x)) \sim A \,\&\, (\exists x)F(x)$.

THEOREM 2′. $\vdash (\exists x)(A \vee F(x)) \sim A \vee (\exists x)F(x)$.

From Theorems 1, 2, 1′ and 2′ rules can be deduced which allow universal and existential quantifiers to be moved across a bracket.

We write out these rules:

(a) $\dfrac{(x)(A \vee B(x))}{A \vee (x)B(x)}$, (a′) $\dfrac{A \vee (x)B(x)}{(x)(A \vee B(x))}$,

(b) $\dfrac{(x)(A \,\&\, B(x))}{A \,\&\, (x)B(x)}$, (b′) $\dfrac{A \,\&\, (x)B(x)}{(x)(A \,\&\, B(x))}$,

$$\text{(c)}\ \frac{(\exists x)(A \,\&\, B(x))}{A \,\&\, (\exists x)B(x)}, \qquad (c')\ \frac{A \,\&\, (\exists x)B(x)}{(\exists x)(A \,\&\, B(x))},$$

$$\text{(d)}\ \frac{(\exists x)(A \vee B(x))}{A \vee (\exists x)B(x)}, \qquad \text{(d}'\text{)}\ \frac{A \vee (\exists x)B(x)}{(\exists x)(A \vee B(x))},$$

under the assumption that the formula A does not contain x in the role of a free variable.

We leave it to the reader to prove the following theorems (making use of the deduction theorem):

$$\vdash (x)A(x) \vee (x)B(x) \to (x)(A(x) \vee B(x)),$$

$$\vdash (x)(A(x) \,\&\, B(x)) \sim (x)A(x) \,\&\, (x)B(x)$$

and the theorems dual to them:

$$\vdash (\exists x)(A(x) \,\&\, B(x)) \to (\exists x)A(x) \,\&\, (\exists x)B(x),$$

$$\vdash (\exists x)(A(x) \vee B(x)) \sim (\exists x)A(x) \vee (\exists x)B(x).$$

To these theorems there correspond the following rules respectively:

$$\frac{(x)A(x) \vee (x)B(x)}{(x)(A(x) \vee B(x))},$$

$$\frac{(x)(A(x) \,\&\, B(x))}{(x)A(x) \,\&\, (x)B(x)}, \qquad \frac{(x)A(x) \,\&\, (x)B(x)}{(x)(A(x) \,\&\, B(x))},$$

$$\frac{(\exists x)(A(x) \,\&\, B(x))}{(\exists x)A(x) \,\&\, (\exists x)B(x)},$$

$$\frac{(\exists x)(A(x) \vee B(x))}{(\exists x)A(x) \vee (\exists x)B(x)}, \qquad \frac{(\exists x)A(x) \vee (\exists x)B(x)}{(\exists x)(A(x) \vee B(x))}.$$

In Chapter III, in the discussion of the algebra of predicates, we established transformations which are analogous to those transformations of equivalence which are expressed in the form of the rules (a)-(d′). Utilizing these transformations, we proved in Chapter III that for every formula there exists a normal formula which is equivalent to it. We can now, starting with the transformations of equivalence (a)-(d′), prove in exactly the same way that for every reduced formula (and, consequently, for an arbitrary formula also) there exists a normal formula which is equivalent to it. The proof can be taken from Chapter III. Although we did not set ourselves there the task of restricting ourselves to constructive methods, none the less the proof was in fact constructive. Therefore, we shall not carry it out in all detail. We recall only that it is based on the fact that one can carry out the transformations of taking quantifiers out of brackets and that it is thus possible to bring all the quantifiers in front of the remaining symbols of the formula.

We shall call a normal formula which is equivalent to a given formula *a normal form of the given formula.*

§13. Deductive equivalence

We now introduce the concept of the "deductive equivalence of formulae". Two formulae A and B are said to be *deductively equivalent* in a calculus if from the axioms of this calculus and the formula A one can deduce, by means of the rules of the calculus, the formula B and, conversely, from the axioms of the calculus and formula B, by means of the rules of the calculus, the formula A is deducible.

For the predicate calculus, the concepts of "equivalent formulae" and "deductively equivalent formulae" are distinct. If A and B are equivalent in the predicate calculus, then they are also deductively equivalent, but not conversely. In fact, suppose A is equivalent to B. This means that the formula $A \sim B$ is true in the predicate calculus. In this case the formula $A \rightarrow B$ is also true. If we adjoin the formula A to the axioms of the predicate calculus, then from the formulae A and $A \rightarrow B$, applying the rule of inference, it is possible to deduce the formula B. Adjoining the formula B to the axioms, one can deduce formula A in a similar manner. It follows from this that *equivalent formulae A and B are also deductively equivalent.* The converse assertion, however, is not true. We consider the elementary formulae A and B. They are deductively equivalent. In fact, if we adjoin the formula A to the axioms of the predicate calculus, then any formula—in particular B—becomes deducible by means of a substitution in the formula A. The same situation holds if we adjoin the formula B to the axioms. It follows from this that the formulae A and B are deductively equivalent in the predicate calculus. However, these formulae are obviously not equivalent since the formula $A \sim B$ is not true in the propositional calculus. Therefore, as we know (see §4), it is also not true in the predicate calculus. We note that for the propositional calculus the concept of deductive equivalence is of little interest. By virtue of the completeness of this calculus in the narrow sense, one of the following two holds for every formula: either it is deducible in the propositional calculus or the adjunction of it to the axioms leads to a contradiction (cf. Chapter II).

We now consider two arbitrary formulae A and B in the propositional calculus. If both of them are deducible in the propositional calculus, then they are simply equivalent. If one of them is deducible and the other is not, then they cannot be deductively equivalent inasmuch as the adjunction of a deducible formula to the axioms does not yield new deducible formulae and therefore the other formula remains non-deducible. If both formulae are non-deducible, then they are deductively equivalent, but then the adjunction of each of them to the axioms forms an inconsistent system.

§14. Skolem's normal forms

Skolem established a very interesting form to which any arbitrary formula in the predicate calculus can be reduced.

A formula is called a *Skolem normal formula* if: firstly, it is in normal form and, secondly, all existential quantifiers—if there are any—precede all the universal quantifiers.

For example, the formulae

$$(\exists x)(\exists y)(z)(u)A(x, y, z, u), \quad (x)(y)A(x, y)$$

are Skolem normal formulae but the formulae

$$(x)(\exists y)A(x, y), \quad (\exists x)(y)(\exists z)A(x, y, z)$$

are not.

SKOLEM'S THEOREM. *For every formula in the predicate calculus there exists a Skolem normal formula which is deductively equivalent to it.*

For the proof of this theorem, we must first prove some lemmas.

LEMMA 1. *The formula* $(\exists x_1) \ldots (\exists x_n)(y)A$, *where* A *is a normal formula, is deductively equivalent to the formula*

$$(\exists x_1) \ldots (\exists x_n)[(y)(A \to A(y)) \to (y)A(y)],$$

where A *is a predicate variable which is not contained in* A.

Proof. To prove this lemma, we shall use the following formulae which were proved above (see §9, Theorems 4 and 5):

$$(x)[A(x) \to B(x)] \to [(x)A(x) \to (x)B(x)], \tag{1}$$

$$(x)[A(x) \to B(x)] \to [(\exists x)A(x) \to (\exists x)B(x)]. \tag{2}$$

We shall assume that the formula

$$(\exists x_1) \ldots (\exists x_n)(y)A(x_1, \ldots, x_n, y) \tag{3}$$

is adjoined, in the role of an axiom, to the predicate calculus or that it is deducible in it. The formula

$$\vdash A(x_1, \ldots, x_n, t) \to ((A \to A(t)) \to A(t))$$

is a deducible formula because—as is easily verified—it is obtained by means of substitutions from a true formula in the propositional calculus. From this formula and from the true formula

$$(y)A(x_1, \ldots, x_n, y) \to A(x_1, \ldots, x_n, t),$$

we obtain, with the aid of the syllogism rule, the formula

$$\vdash (y)A(x_1, \ldots, x_n, y) \to ((A \to A(t)) \to A(t)).$$

And furthermore, applying the first rule for binding by a quantifier and renaming the variables, we obtain

$$\vdash (y)A(x_1, \ldots, x_n, y) \to (y)((A \to A(y)) \to A(y)).$$

Next, on the basis of formula (1), utilizing the syllogism rule, we obtain

$$\vdash (y)A(x_1, \ldots, x_n, y) \to [(y)(A \to A(y)) \to (y)A(y)].$$

Applying to this formula the derived rule for binding by a quantifier, we obtain

$$\vdash (x_n)[(y)A(x_1, \ldots, x_n, y) \rightarrow ((y)(A \rightarrow A(y)) \rightarrow (y)A(y))]. \tag{4}$$

Performing a substitution in formula (2) in such a way that the predicate $A(t)$ is replaced by the formula

$$(y)A(x_1, \ldots, x_{n-1}, t, y)$$

and the predicate $B(t)$ by the formula

$$(y)A(x_1, \ldots, x_{n-1}, t, y) \rightarrow A(y)) \rightarrow (y)A(y),$$

we obtain

$$\begin{aligned}\vdash (x_n)[(y)A(x_1, \ldots, x_n, y) \rightarrow ((y)(A(x_1, \ldots, x_n, y) \rightarrow A(y)) \rightarrow (y)A(y)]\\ \rightarrow [(\exists x_n)(y)A(x_1, \ldots, x_n, y) \rightarrow\\ \rightarrow (\exists x_n)((y)(A(x_1, \ldots, x_n, y) \rightarrow A(y)) \rightarrow (y)A(y))].\end{aligned}$$

The antecedent in this formula is the deducible formula (4). Therefore, on the basis of the rule of inference, the consequent is also a true formula, i.e.

$$\begin{aligned}\vdash (\exists x_n)(y)A(x_1, \ldots, x_n, y) \rightarrow (\exists x_n)((y)(A(x_1, \ldots, x_n, y) \rightarrow\\ \rightarrow A(y)) \rightarrow (y)A(y)).\end{aligned}$$

In an analogous manner, from this formula, binding it with the quantifier (x_{n-1}), we obtain, with the aid of formula (2),

$$\begin{aligned}\vdash (\exists x_{n-1})(\exists x_n)(y)A(x_1, x_2, \ldots, x_n, y) \rightarrow\\ \rightarrow (\exists x_{n-1})(\exists x_n)[(y)(A(x_1, \ldots, x_n, y) \rightarrow A(y)) \rightarrow (y)A(y)].\end{aligned}$$

Binding in turn with the quantifiers (x_{n-2}), (x_{n-3}), ... and proceeding in the same way we arrive at the formula

$$\begin{aligned}\vdash (\exists x_1) \ldots (\exists x_n)(y)A(x_1, \ldots, x_n, y) \rightarrow\\ \rightarrow (\exists x_1) \ldots (\exists_n)[(y)(A(x_1, \ldots, x_n, y) \rightarrow A(y)) \rightarrow (y)A(y)]. \quad (*)\end{aligned}$$

From this it follows that the formula

$$(\exists x_1) \ldots (\exists x_n)[(y)(A(x_1, \ldots, x_n, y) \rightarrow A(y)) \rightarrow (y)A(y)] \tag{5}$$

is deducible from the antecedent in (*), above, i.e. from formula (3).

We have thus proved that if we adjoin formula (3) in the role of an axiom to the axioms of the predicate calculus, then formula (5) becomes deducible. By the same token, the lemma is proved in one direction.

We shall now assume that formula (5) has been adjoined to the axioms of the predicate calculus. We make a substitution in it, replacing the predicate $A(t)$ by the formula $A(x_1, \ldots, x_n, t)$. We obtain the following formula (which is deducible in the new system):

$$\begin{aligned}(\exists x_1) \ldots (\exists x_n)[(y)(A(x_1, \ldots, x_n, y) \rightarrow\\ \rightarrow A(x_1, \ldots, x_n, y)) \rightarrow (y)A(x_1, \ldots, x_n, y)].\end{aligned} \tag{6}$$

The formula

$$(T \rightarrow (y)A(y)) \rightarrow (y)A(y),$$

where T is an arbitrary true formula, is also true inasmuch as it is obtained by a substitution in a true formula of the propositional calculus. Since $(y)(A \to A)$ is a true formula, the formula

$$((y)(A \to A) \to (y)A(y)) \to (y)A(y)$$

is also true. Binding this formula with the quantifier (x_n), we obtain the true formula

$$(x_n)[((y)(A \to A) \to (y)A(y)) \to (y)A(y)].$$

Applying to this formula the same line of reasoning as in the first part of the proof, we obtain the true formula

$$(\exists x_n)[(y)(A \to A) \to (y)A(y)] \to (\exists x_n)(y)A(y).$$

Next, binding this formula successively by the quantifiers $(x_{n-1}), \ldots, (x_1)$ and repeating the same line of reasoning, we finally obtain

$$\vdash (\exists x_1) \ldots (\exists x_n)[(y)(A \to A) \to (y)A(y)] \to (\exists x_1) \ldots (\exists x_n)(y)A(y).$$

The antecedent of this implication is formula (6) which is deducible in the system of axioms of the predicate calculus after the adjunction of formula (5) to it. And, then, on the basis of the rule of inference the consequent—i.e. formula (3)—is also deducible after the adjunction of formula (5). The deductive equivalence of formulae (3) and (5) is thus proved.

We shall assume that the formula A has the form

$$(Q_1z_1)(Q_2z_2) \ldots (Q_mz_m)C,$$

where (Q_iz_i) is a general designation for the quantifiers (z_i) and $(\exists z_i)$, and C does not contain quantifiers. [The quantifiers (Q_iz_i) may be completely absent.]

LEMMA 2. *For the formula A indicated above we have:*

$$\vdash [(A(x_1, \ldots, x_n, y) \to A(y)) \to (z)A(z)] \sim (Q_1z_1) \ldots$$
$$\ldots (Q_mz_m)(z)[(C(z_1, \ldots, z_m, x_1, \ldots, x_n, y) \to A(y)) \to A(z)]. \quad (7)$$

In order to prove this equivalence, we transform its left member. We first write in the quantifiers in A. We obtain

$$((Q_1z_1) \ldots (Q_mz_m)C \to A(y)) \to (z)A(z).$$

We next delete the sign $\to$ by means of known equivalent transformations; we obtain

$$\overline{\overline{(Q_1z_1) \ldots (Q_mz_m)C} \vee A(y)} \vee (z)A(z).$$

After this, we transform the left member, bringing the upper negation sign into the interior of the sum, and we delete the double negation obtained. We then have

$$(Q_1z_1) \ldots (Q_mz_m)C \,\&\, \bar{A}(y) \vee (z)A(z).$$

We next bring the quantifiers $(Q_1 z_1), \ldots, (Q_m z_m)$ and (z) outside the brackets —which, as we already know, can be done (see §12). We obtain

$$(Q_1 z_1) \ldots (Q_m z_m)(z)(C \,\&\, \bar{A}(y) \vee A(z)).$$

The formula appearing after all the quantifiers $(Q_i z_i)$ and (z) is equivalent to the formula

$$(C \to A(y)) \to A(z).$$

Replacing it by this equivalent formula, we have

$$(Q_1 z_1) \ldots (Q_m z_m)(z)[(C \to A(y)) \to A(z)].$$

This formula is the right member of the equivalence (7) which we are required to prove. But since we obtained it by means of equivalent transformations from the left member of equivalence (7), then this equivalence actually holds—and the lemma is proved.

LEMMA 3. *The formula*

$$(\exists x_1) \ldots (\exists x_n)[(y)(A \to A(y)) \to (z)A(z)] \tag{8}$$

is equivalent to the formula

$$(\exists x_1) \ldots (\exists x_n)(\exists y)(Q_1 z_1) \ldots (Q_m z_m)(z)[(C(z_1, \ldots, z_n, y) \to \to A(y)) \to A(z)]. \tag{9}$$

Proof. We can conclude from Lemma 2 that

$$\vdash [(A(x_1, \ldots, x_n, y) \to A(y) \to (z)A(z)] \to (Q_1 z_1) \ldots \ldots (Q_m z_m)(z)[(C(z_1, \ldots, z_n, x_1, \ldots, x_n, y) \to A(y)) \to A(z)]$$

holds. Binding this true formula by a quantifier, we obtain

$$\vdash (y)\{[(A(x_1, \ldots, x_n, y) \to A(y)) \to (z)A(z)] \to (Q_1 z_1) \ldots \ldots (Q_m z_m)(z)[(C(z_1, \ldots, z_m, x_1, \ldots, x_n, y) \to A(y)) \to A(z)]\}.$$

Utilizing formula (2) as this was done in Lemma 1, we find that

$$\vdash (\exists y)[(A(x_1, \ldots, x_n, y) \to A(y)) \to (z)A(z)] \to (\exists y)(Q_1 z_1) \ldots \ldots (Q_m z_m)(z)[(C(z_1, \ldots, z_m, x_1, \ldots, x_n, y) \to A(y)) \to A(z)]. \tag{10}$$

Considering the reverse implication which follows from Lemma 2:

$$\vdash (Q_1 z_1) \ldots (Q_m z_m)(z)[(C(z_1, \ldots, z_m, x_1, \ldots, x_n, y) \to \to A(y)) \to A(z)] \to [(A(x_1, \ldots, x_n, y) \to A(y)) \to (z)A(z)],$$

we obtain in the same way that

$$\vdash (\exists y)(Q_1 z_1) \ldots (Q_m z_m)(z)[(C(z_1, \ldots, z_m, x_1, \ldots, x_n, y) \to A(y)) \to \to A(z)] \to (\exists y)[(A(x_1, \ldots, x_n, y) \to A(y)) \to (z)A(z)]. \tag{11}$$

Combining the mutually-reverse implications (10) and (11), we obtain

$$\vdash (\exists y)[(A(x_1, \ldots, x_n, y) \to A(y)) \to (z)A(z)] \sim (\exists y)(Q_1 z_1) \ldots \ldots (Q_m z_m)(z)[(C(z_1, \ldots, z_m, x_1, \ldots, x_n, y) \to A(y)) \to A(z)].$$

Clearly, starting with this formula, we can repeat our line of reasoning for the quantifiers $(\exists x_n)$, $(\exists x_{n-1})$, ..., $(\exists x_1)$. As a result, we arrive at the equivalence

$$\vdash (\exists x_1) \dots (\exists x_n)(\exists y)[(A(x_1, \dots, x_n, y) \to A(y)) \to (z)A(z)] \sim$$
$$\sim (\exists x_1) \dots (\exists x_n)(\exists y)(Q_1 z_1) \dots (Q_m z_m)(z)$$
$$[(C(z_1, \dots, z_m, x_1, \dots, x_n, y) \to A(y)) \to A(z)]. \quad (12)$$

We shall show that the following equivalence holds:

$$\vdash [(y)(A \to A(y)) \to (z)A(z)] \sim (\exists y)[(A \to A(y)) \to (z)A(z)]. \quad (13)$$

In order to prove the truth of this equivalence, it suffices, by means of equivalent transformations, to obtain its right member from its left member. To this end, we first exclude the sign $\to$ from the left member; we obtain the formula

$$\overline{(y)(\bar{A} \vee A(y))} \vee (z)A(z).$$

Bringing the negation sign under the sign of the quantifier (y), we obtain

$$(\exists y)\overline{(\bar{A} \vee A(y))} \vee (z)A(z).$$

Factoring out the quantifier $(\exists y)$, we have

$$(\exists y)[\overline{\bar{A} \vee A(y)} \vee (z)A(z)].$$

Introducing the sign $\to$ again, we obtain the left member of equivalence (13):

$$(\exists y)[(A \to A(y)) \to (z)A(z)].$$

The truth of the required equivalence is thus proved.

Applying to this equivalence the same line of reasoning which we have already made use of many times in this section, we obtain

$$\vdash (\exists x_1) \dots (\exists x_n)[(y)(A \to A(y)) \to (z)A(z)] \sim$$
$$\sim (\exists x_1) \dots (\exists x_n)(\exists y)[(A \to A(y)) \to (z)A(z)]. \quad (14)$$

Considering equivalences (14) and (12), we obtain, with the aid of the syllogism rule, the required equivalence of formulae (8) and (9), and the lemma is thus proved.

§15. Proof of Skolem's theorem

We can deduce the following conclusion from Lemmas 1 and 3.

The formula $(\exists x_1) \dots (\exists x_n)(y)A(x_1, \dots, x_n, y)$ *or, what amounts to the same thing, the formula*

$$(\exists x_1) \dots (\exists x_n)(y)(Q_1 z_1) \dots (Q_m z_m)C(z_1, \dots, z_m, x_1, \dots, x_n, y) \quad (15)$$

is deductively equivalent to the formula

$$(\exists x_1) \dots (\exists x_n)(\exists y)(Q_1 z_1) \dots (Q_m z_m)(z)C_1, \quad (16)$$

where C_1 *is the formula*

$$(C \to A(y)) \to A(z).$$

C and C_1 do not contain quantifiers. The first universal quantifier (in

order) in formula (15) was replaced in the deductively equivalent formula (16) by the existential quantifier, and moreover a new universal quantifier appeared in (16)—the last one (in order). If, in formula (16), there is a universal quantifier among the quantifiers $(Q_i z_i)$, then, applying the same line of reasoning to it, we can obtain a formula which is deductively equivalent to it and, consequently, also to formula (15), in which the first of the universal quantifiers $(Q_r z_r)$ was replaced by an existential quantifier and a new one again appeared, as the last universal quantifier (in order). If formula (16) has the form

$$(\exists x_1) \ldots (\exists x_n)(\exists y)(z_1) \ldots (\exists z_{r-1})(z_r)(Q_{r+1}z_{r+1}) \ldots (Q_m z_m)(z)C_1,$$

then the formula deductively equivalent to it has the form

$$(\exists x_1) \ldots (\exists x_n)(\exists y)(\exists z_1) \ldots (\exists z_r)(Q_{r+1}z_{r+1}) \ldots (Q_m z_m)(z)(z')C_2.$$

Continuing this process further we finally obtain the formula

$$(\exists x_1) \ldots (\exists x_n)(\exists y)(\exists z_1) \ldots (\exists z_m)(z)(z') \ldots (z^{(p-1)})C_p, \tag{17}$$

which is deductively equivalent to formula (15), i.e. to the formula

$$(\exists x_1) \ldots (\exists x_n)(y)A. \tag{3}$$

Moreover, (17) is the Skolem normal formula. The theorem will be proved if we show that every formula is deductively equivalent to a formula of the form (15).

We know that for every formula there exists a normal formula which is equivalent to it. It is therefore sufficient to prove that every normal formula is equivalent to a formula of form (15).

Let us consider an arbitrary normal formula:

$$(Q_1 z_1) \ldots (Q_m z_m)B, \tag{18}$$

where B does not contain quantifiers. Of course, there might not be any quantifiers $(Q_i z_i)$ at all. Let x and y be variables which are not contained in this formula. The formula

$$(\exists x)(y)(Q_1 z_1) \ldots (Q_m z_m)[B \,\&\, (A(x) \vee \bar{A}(x)) \,\&\, (A(y) \vee \bar{A}(y))] \tag{19}$$

is a formula of the form (15). If in it all quantifiers $(Q_i z_i)$ are brought into the brackets, we obtain the equivalent formula

$$(\exists x)(y)[(Q_1 z_1) \ldots (Q_m z_m)B \,\&\, (A(x) \vee \bar{A}(x)) \,\&\, (A(y) \vee \bar{A}(y))].$$

It is easily seen that this formula is equivalent to formula (18) inasmuch as for an arbitrary formula A which does not contain x and y, the equivalance

$$\vdash (\exists x)(y)[A \,\&\, (A(x) \vee \bar{A}(x)) \,\&\, (A(y) \vee \bar{A}(y))] \sim A$$

holds.

The proof of this equivalence is very straightforward, and so we shall leave it to the reader. As a result of this equivalence, we find that an arbitrary normal formula (18) is equivalent to formula (19) of the form (15).

And we proved above that this suffices for the proof of Skolem's theorem. We have thus proved that for every formula there exists a Skolem normal formula which is deductively equivalent to it. We shall call the Skolem normal formula which is deductively equivalent to a given formula A its *Skolem normal form.*

§16. Maltsev's theorem

The formal treatment of the propositional calculus or of propositional algebra allows one to generalize the definition of the logical sum and logical product to the case of an infinite number of logical variables and in this way introduce infinite formulae.

We shall define logical concepts inductively, starting from elementary propositional variables—of which there can now exist an infinite set of arbitrary power. These elementary variables can take two, and only two, values: T and F. We shall denote these variables either by upper-case Latin letters, as before:

$$A, B, C, \ldots,$$

or by letters with indices:

$$A_\xi, X_\delta, \ldots, P_n, \ldots,$$

where the indices can take on values from an arbitrary set of objects. Suppose given some set of formulae, which are already defined and represent functions of the variables occurring in them. We denote this set by $\{A\}$, where A is the symbol for an arbitrary element of the given set of formulae.

The logical product ΠA is a formula which takes on the value T if, and only if, all A take on the value T. Consequently, ΠA takes on the value F if at least one of the formulae A takes on the value F.

The logical sum ΣA is defined analogously: the formula ΣA is true if, and only if, at least one of the formulae A is true.

In the case when $\{A\}$ is a countable set of formulae and can consequently be represented in the form of a sequence of elements which are indexed by natural numbers:

$$A_1, A_2, \ldots, A_n, \ldots,$$

the formula ΠA_n can be written as

$$A_1 \& A_2 \& \ldots \& A_n \& \ldots$$

and the formula ΣA_n as

$$A_1 \vee A_2 \vee \ldots \vee A_n \vee \ldots.$$

The duality law for infinite formulae is the same as for finite formulae:

$$\overline{\Pi A} \text{ is equivalent to } \Sigma \bar{A};$$

$$\overline{\Sigma A} \text{ is equivalent to } \Pi \bar{A}.$$

We shall not stop to prove these propositions because they are obtained by a verbatim repetition of the proof, introduced above, of the duality law in the propositional calculus.

In exactly the same way as above, making use of this law, we can replace all operations by two operations: & and — (or by $\vee$ and —). The Soviet mathematician A. I. Maltsev proved an interesting theorem interrelating certain infinite formulae of the propositional calculus with finite formulae of the propositional calculus, a theorem having far-reaching applications.

MALTSEV'S THEOREM. *Let ΣA be an arbitrary logical sum all terms A of which are finite formulae. If ΣA is an identically true formula, then a finite number N of its summands can be found the sum of which $A_1 \vee A_2 \vee \ldots \vee A_N$ is also identically true.*

This theorem is valid for an arbitrary set of finite logical summands. However, to prove it in the most general form necessitates use of transfinite numbers. We shall carry out the proof of this theorem under the assumption that the set of formulae A is countable. Such a result will suffice us in subsequent applications. Thus, we are given a countable logical sum of finite formulae:

$$A_1 \vee A_2 \vee \ldots \vee A_n \vee \ldots .$$

By the condition of the theorem, it is identically true. We shall prove that there then exists a natural number N such that the sum

$$A_1 \vee A_2 \vee \ldots \vee A_N$$

is an identically true formula. Let us assume the contrary; then none of the sums $\sum\limits_{i=1}^{n} A_i$ is identically true. Consequently, one can find values of the logical variables occurring in the sum $\sum\limits_{i=1}^{n} A_i$ for which it takes on the value F. Let

$$A_1^n, A_2^n, \ldots, A_{k_n}^n \tag{1}$$

be the variables occurring in the formula $\sum\limits_{i=1}^{n} A_i$ and let

$$a_1^n, a_2^n, \ldots, a_{k_n}^n \tag{2}$$

be those values of these variables for which it takes on the value F. Each a_i^n is either T or F. The number of all possible arrangements of the values which the variables (1) can take on equals 2^{k_n}, i.e. a finite number. Consider

$$A_1(A_1^1, A_2^1, \ldots, A_{k_1}^1).$$

All the variables A_i^1 are contained in every formula $\sum\limits_{i=1}^{n} A_i$ and they consequently occur among the variables (1) whatever n is. We shall denote them by

$$A_1^{1,n}, A_2^{1,n}, \ldots, A_{k_1}^{1,n}.$$

Thus, $A_i^{1,n}$, $i \leqslant k_n$, is nothing else than A_i^1. The values which the variables $A_i^{1,n}$ take on in the set of variables (1) and which we denoted by a_i^n will now be denoted by $a_i^{1,n}$. Although $A_i^{1,n}$ coincides with the variable A_i^1, the value

$\alpha_i^{1,n}$ will, in general, not coincide with the value α_i^1. In fact, substituting in the formula $\sum_{i=1}^{n} A_i$ the values of the variables which make this formula take on the value F, we have no grounds for thinking that the variables $A_1^1, \ldots, A_{k_1}^1$ of the formula A_1, variables which also occur in the formula $\sum_{i=1}^{n} A_i$, take on the very same values $\alpha_1^1, \ldots, \alpha_{k_1}^1$ which were chosen for the formula A_1. In general, if $n < m$, then the variables of the formula $\sum_{i=1}^{n} A_i$ which occur among the variables of the formula $\sum_{i=1}^{m} A_i$ will also be denoted by

$$A_1^{n,m}, A_2^{n,m}, \ldots, A_{k_n}^{n,m},$$

and their values in the set $(\alpha_1^m, \ldots, \alpha_{k_m}^m)$ by

$$\alpha_1^{n,m}, \alpha_2^{n,m}, \ldots, \alpha_{k_n}^{n,m}.$$

The formula $\sum_{i=1}^{n} A_i$ is false for the values (2) of the variables; consequently, every summand of this formula is also false for these values. Consequently, formula A_1 must be false for the values

$$\alpha_1^{1,n}, \alpha_2^{1,n}, \ldots, \alpha_{k_1}^{1,n}$$

of the variables—whatever n is.

We now consider the sequence of sets of values of k_1 variables occurring in the formula A_1:

$$(\alpha_1^{1,1}, \ldots, \alpha_{k_1}^{1,1}), (\alpha_1^{1,2}, \ldots, \alpha_{k_1}^{1,2}), \ldots, (\alpha_1^{1,n}, \ldots, \alpha_{k_1}^{1,n}), \ldots. \tag{3}$$

There can be only a finite number (not greater than 2^{k_1}) distinct sets in this sequence. Therefore, one of these sets is repeated an infinite number of times in sequence (3). We can consequently select from sequence (3) an infinite subsequence of identical sets:

$$(\alpha_1^{1,n_1}, \ldots, \alpha_{k_1}^{1,n_1}), (\alpha_1^{1,n_2}, \ldots, \alpha_{k_1}^{1,n_2}), \ldots, (\alpha_1^{1,n_j}, \ldots, \alpha_{k_1}^{1,n_j}), \ldots, \tag{a$_1$}$$

where $\alpha_i^{1,n_r} = \alpha_i^{1,n_j}$ for arbitrary r and j. For values of the variables A_i^1 equal to α_i^{1,n_j}, the formula A_1 takes on the value F.

Moreover, we have an infinite sequence of formulae:

$$\sum_{i=1}^{n_1} A_i, \sum_{i=1}^{n_2} A_i, \ldots, \sum_{i=1}^{n_j} A_i, \ldots,$$

where each formula $\sum_{i=1}^{n_j} A_i$ takes on the value F when the variables occurring in it take on the values

$$\alpha_1^{n_j}, \alpha_2^{n_j}, \ldots, \alpha_{k_{n_j}}^{n_j}.$$

Upon replacing the variables in the formula $\sum_{i=1}^{n_j} A_i$ by these values, every variable A_i^1 takes on the same value whatever n_j is. But the remaining

propositional variables occurring in $\sum_{i=1}^{n_j} A_i$ and not occurring in A_1 assume, in general, distinct values $a_i^{n_j}$ for distinct n_j.

Consider the formula $\sum_{i=1}^{2} A_i$ or, what amounts to the same thing, $A_1 \vee A_2$. For every $n_j \geqslant 2$, the variables of this formula occur among the variables of the formula $\sum_{i=1}^{n_j} A_i$. The sets of values of the variables A_i^2 occurring in $\sum_{i=1}^{n_j} A_i (j = 1, 2, \ldots)$ for which these formulae take on the value F form an infinite sequence:

$$(a_1^{2, n_1}, \ldots, a_{k_2}^{2, n_1}), (a_1^{2, n_2}, \ldots, a_{k_2}^{2, n_2}), \ldots, (a_1^{2, n_j}, \ldots, a_{k_2}^{2, n_j}), \ldots, \tag{4}$$

where the values of each variable occurring in A_1 are the same in all sets of sequence (4). They coincide with the values in sequence (a_1).

Reasoning as for the case $n = 1$, we can select from sequence (4) a subsequence of identical sets:

$$(a_1^{2, m_1}, \ldots, a_{k_2}^{2, m_1}), (a_1^{2, m_2}, \ldots, a_{k_2}^{2, m_2}), \ldots, (a_1^{2, m_j}, \ldots, a_{k_2}^{2, m_j}), \ldots, \tag{a_2}$$

where

$$a_i^{2, m_j} = a_i^{2, m_k} \text{ for } i = 1, 2, \ldots, k_2;$$
$$j, k = 1, 2, \ldots, n, \ldots; \quad m_j, m_k \geqslant 2.$$

Every set of values of the variables A_i^2 of the sequence (a_2) transforms the formula $A_1 \vee A_2$ into F. The values of the variables A_i^2, occurring in A_1, in the sequence (a_2) are the same as in sequence (a_1). Reasoning in the same way further, we obtain the following countable set of sequences:

$$(a_1^{3, p_1}, \ldots, a_{k_3}^{3, p_1}), (a_1^{3, p_2}, \ldots, a_{k_3}^{3, p_2}), \ldots, (a_1^{3, p_j}, \ldots, a_{k_3}^{3, p_j}), \ldots \tag{a_3}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a_1^{n, q_1}, \ldots, a_{k_n}^{n, q_1}), (a_1^{n, q_2}, \ldots, a_{k_n}^{n, q_2}), \ldots, (a_1^{n, q_j}, \ldots, a_{k_n}^{n, q_j}), \ldots \tag{a_n}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

where $a_i^{n, q_j} = a_i^{n, q_r}$. The sets occurring in the sequence (a_n) are the sets of values of all variables occurring in the formula $\sum_{i=1}^{n} A_i$. They transform this formula into F.

We now consider the following sequence:

$$(a_1^{1, n_1}, \ldots, a_{k_1}^{1, n_1}), (a_1^{2, m_1}, \ldots, a_{k_2}^{2, m_1}), \ldots, (a_1^{n, q_1}, \ldots, a_{k_n}^{n, q_1}), \ldots. \tag{b}$$

The first set consists of values of the variables A_i^1 which transform A_1 into F, the second of values of the variables A_i^2 which transform $A_1 \vee A_2$ into F, and so on.

These sets possess the following property: *Whatever the variable* X_i

occurring in some formula $\sum_{i=1}^{n} A_i$, *its value is the same in all sets of the sequence* (b) *in which it occurs*. In fact, suppose the variable X occurs in the formula $\sum_{i=1}^{n} A_i$ for certain of the n. Let n_0 be the smallest value of n for which X occurs in such a formula. Then, for every $n \geqslant n_0$, the variable X coincides with some $A_r^{n_0, n}$, i.e. $A_r^{n_0, n}$ is the designation of the variable X in the formula $\sum_{i=1}^{n} A_i$. By construction, the values $\alpha_r^{n_0, n}$ of the variable $A_r^{n_0, n}$ in all sets of the sequence (a_{n_0}) are the same. But the value of the variable $A_r^{n_0, n}$, occurring in $\sum_{i=1}^{n_0} A_i$, is the same in all sets of the sequence (a_n) for $n > n_0$ as in the sets of the sequence (a_{n_0}). Consequently, the value of the variable $A_r^{n_0, n}$, i.e. X, is also the same in all sets of the sequence (b).

To each variable occurring in some formula $\sum_{i=1}^{n} A_i$ we set into correspondence, by means of the sequence (b), some one of its values.

The sets of the sequence (b) consist of the values of all variables which occur in the formulae:

$$A_1, A_1 \vee A_2, \ldots, A_1 \vee A_2 \vee \ldots \vee A_n, \ldots, \tag{5}$$

respectively. The set of the sequence (b) with index n thus consists of the values of all variables occurring in $\sum_{i=1}^{n} A_i$. Therefore, if the variable X occurs in $\sum_{i=1}^{n} A_i$, then its value appears in the set with index n. As we saw, the value of X is the same in all sets in which it occurs. And it is this value which we shall put into correspondence with the variable X.

We shall give to each variable occurring in the infinite formula

$$A_1 \vee A_2 \vee \ldots \vee A_n \vee \ldots \tag{6}$$

that value which was set into correspondence with it by means of sequence (b). These values are such that formulae (5) takes on the value F upon replacement of every variable by the value corresponding to it. But then, whatever n is, the formula A_n also takes on the value F upon the replacement of all variables by the values corresponding to them. Thus, every summand of sum (6) takes on the value F for a given substitution of values of the variables. But then the sum (6) itself also takes on the value F and, hence, it is not identically true which contradicts the condition of the theorem. Thus, the assumption that all the sums $\sum_{i=1}^{N} A_i$ are not identically true formulae leads to a contradiction. Consequently, there exists at least one finite identically true sum $\sum_{i=1}^{N} A_i$, which is what we were required to prove.

The theorem just proved can be formulated as follows:

If all finite products $\prod_{i=1}^{N} A_i$ *which are factors of the infinite product* $\prod_{n=1}^{\infty} A_n$ *of finite formulae* A_i *are satisfiable, then* $\prod_{n=1}^{\infty} A_n$ *is also a satisfiable formula.*

In fact, if $\prod_{n=1}^{\infty} A_n$ turned out to be a non-satisfiable formula, then $\sum_{n=1}^{\infty} \bar{A}_n$ would be an identically true formula. But then, by virtue of the theorem just proved, there exists an N such that the sum $\sum_{n=1}^{N} \bar{A}_n$ is identically true. And then the negation of this formula—which is equivalent to the formula $\prod_{n=1}^{N} A_n$—would be a non-satisfiable formula, which contradicts the hypothesis.

§17. Completeness problem for the predicate calculus in the wide sense

In our consideration of predicate logic from the informal point of view (cf. Chapter III), we introduced the concept of an identically true formula which corresponds—as far as its meaning goes—to the concept of a "tautologically true proposition". On the other hand, in the predicate calculus we also have the concept of a true or deducible formula. The question arises of the relation of these two concepts. As we pointed out above, every formula which is deducible in the predicate calculus is also identically true in the informal sense (see §4).

Conversely, *is every identically true formula deducible in the predicate calculus?* This question is called the *completeness problem for the predicate calculus in the wide sense.* We shall see in §19 that the completeness problem in the wide sense is solved in the affirmative. One must, however, note that in the solution of the completeness problem of the predicate calculus in the wide sense we cannot restrict ourselves to the methods of reasoning of finitary metalogic in view of the fact that the concept of an "identically true formula" occurs in the very formulation of the problem, a concept which involves the consideration of *all* interpretations.

We make one further remark of a technical kind. *If two formulae are deductively equivalent, then it follows from the fact that one of them is identically true that the other one is also identically true.* In fact, let A and B be deductively equivalent formulae and suppose A is an identically true formula. As we saw, all formulae which are deducible from identically true formulae are also identically true formulae. In virtue of the deductive equivalence, B is deducible from the axioms of the predicate calculus and the formula A; therefore, B is also an identically true formula. Moreover, if A and B are deductively equivalent and A is a formula which is deducible in the predicate calculus, then B is also deducible in the predicate calculus. The last fact follows from the definition of deductive equivalence.

It follows from everything we stated above that in the solution of the completeness problem of the predicate calculus in the wide sense, we can restrict ourselves to the consideration of only Skolem normal formulae. In fact, let us assume that we have proved that every identically true Skolem normal formula is deducible in the predicate calculus. Let A be an arbitrary formula and suppose A^* is its Skolem normal form. If A is identically true, then A^* is also identically true. But then A^* is deducible in the predicate calculus and, consequently, the formula A which is deductively equivalent to it is also deducible in the predicate calculus.

§18. Remarks about formulae of the predicate calculus which do not contain quantifiers

A formula of the predicate calculus which does not contain quantifiers can be considered in a certain sense as a formula of the propositional calculus and as a formula of propositional algebra. To show this, it is only necessary to consider the predicate variables occurring in the formula as propositional variables—simply denoted in a special way. We shall assume that such propositional variables, represented by predicate variables, are identical if, and only if, the predicates expressing them are completely identical, i.e. they have precisely the same transcription.

EXAMPLES:

1. We shall think of the formula

$$F(x) \to F(x) \,\&\, F(y)$$

as a formula of the propositional calculus, where $F(x)$ and $F(y)$ are *distinct* propositional variables. In the usual notation, such a formula would have the form

$$A \to A \,\&\, B.$$

2. $(F(x, y) \to F(x, y) \vee (\bar{F}(y, y) \to \bar{F}(x, y))$.

We could give this formula—considered as a formula of propositional algebra—the following form:

$$(A \to B) \vee (\bar{C} \to \bar{A}).$$

3. $A \vee F(x) \,\&\, B \vee \bar{F}(y)$.

This formula, considered as a formula in propositional algebra has the form

$$(A \vee C) \,\&\, (B \vee D).$$

Let A be a formula of the predicate calculus which does not contain quantifiers. We shall assume that this formula—considered as a formula of propositional algebra—is satisfiable. This means that it assumes the value T for certain values of the propositional variables. We can then replace the propositional variables and predicate variables by the values T and F—

retaining the condition that identical propositions and predicates are replaced in the same way—in such a way that a formula assuming the value T is obtained. In fact, returning to the consideration of this formula as a formula of the predicate calculus, we can assert that:

Replacing the object variables of a formula by the objects of an arbitrary field under the condition that distinct variables are replaced by distinct objects, one can find predicates defined on this field for which the formula obtained takes on the value T.

The choice of the field is restricted only by the fact that the number of objects it contains must not be less than the number of distinct object variables occurring in the formula. Thus, an arbitrary field, containing an infinite set of objects, can be utilized in the indicated sense for an arbitrary formula which does not contain quantifiers.

If a formula A, considered as a formula in propositional algebra, is identically true, then—as a formula in the propositional calculus—it is deducible in the propositional calculus. But then this formula, again considered as a formula of the predicate calculus, is deducible in the predicate calculus also inasmuch as it is obtained by substitutions in a true formula of the propositional calculus.

In the sequel, for the sake of brevity, instead of saying "a formula A—considered as a formula in the propositional calculus—is deducible in the propositional calculus", we shall simply say "a formula A which is deducible in the propositional calculus".

§19. Gödel's completeness theorem

GÖDEL'S THEOREM. *Every identically true formula in predicate logic is deducible in the predicate calculus.*

To prove this theorem, we introduce certain notations and prove a preliminary lemma. It follows from Skolem's theorem and the remarks made in §17 that to prove Gödel's theorem it is sufficient to consider only Skolem normal formulae.

Let A be a Skolem normal formula. It then has the form

$$(\exists x_1) \ldots (\exists x_k)(y_1) \ldots (y_m)M(x_1, \ldots, x_k, y_1, \ldots, y_m). \tag{1}$$

We introduce the following sequence of new symbols for object variables:

$$t_0, t_1, \ldots, t_n, \ldots .$$

We consider all possible sets $(t_{i_1}, \ldots, t_{i_k})$ of k symbols of this sequence and arrange these sets in a sequence in such a way that the sets with greatest sum of indices, $i_1 + i_2 + \ldots + i_k$, follow after the sets with a smaller sum of indices, and sets with the same sum of indices are ordered arbitrarily, for instance, lexicographically:

$$(t_0, t_0, \ldots, t_0), (t_0, \ldots, t_0, t_1), (t_0, \ldots, t_1, t_0), \ldots, (t_{i_1}, \ldots, t_{i_k}), \ldots .$$

Suppose the jth set of this sequence is $(t_{i_1}, t_{i_2}, \ldots, t_{i_k})$. We introduce the following notation:

$$\left.\begin{array}{l} \boldsymbol{B}_j \text{ denotes the formula} \\ \boldsymbol{M}(t_{i_1}, \ldots, t_{i_k}, t_{(j-1)m+1}, t_{(j-1)m+2}, \ldots, t_{jm}) \\ \boldsymbol{C}_j \text{ denotes the formula } \boldsymbol{B}_1 \vee \boldsymbol{B}_2 \vee \ldots \vee \boldsymbol{B}_j \\ \boldsymbol{D}_j \text{ denotes the formula } (t_0) \ldots (t_{jm})\boldsymbol{C}_j. \end{array}\right\} \quad (2)$$

We write out the formula $\boldsymbol{D}_j$ in more detail:

$$\begin{aligned}(t_0) \ldots (t_{jm})(\boldsymbol{B}_1(t_0, \ldots, t_0, t_1, \ldots, t_m) \vee \\ \vee \boldsymbol{B}_2(t_0, \ldots, t_1, t_{m+1}, \ldots, t_{2m}) \vee \ldots \\ \ldots \vee \boldsymbol{B}_j(t_{i_1}, \ldots, t_{i_k}, t_{(j-1)m+1}, \ldots, t_{jm})). \qquad (3)\end{aligned}$$

It is easily seen that with the given method of enumerating the set $t_{i_1}, \ldots, t_{i_k}$ the index $(j-1)m + 1$ exceeds the indices $i_1, \ldots, i_k$ of the elements of the jth set. In fact, the index of every element occurring in the set is less than the index of the set. Thus, the formula

$$\boldsymbol{C}_{j-1}, \text{ i.e. } \boldsymbol{B}_1 \vee \boldsymbol{B}_2 \vee \ldots \vee \boldsymbol{B}_{j-1}$$

does not contain the variables $t_{(j-1)m+1}, \ldots, t_{jm}$, and the formula $\boldsymbol{C}_j$ contains only the variables $t_0, t_1, \ldots, t_{jm}$. On the other hand, $\boldsymbol{C}_j$ contains all of these variables inasmuch as $\boldsymbol{B}_1$ contains the variables $t_0, t_1, \ldots, t_m$; $\boldsymbol{B}_2$ contains the variables $t_{m+1}, \ldots, t_{2m}$; and so on, where, finally, $\boldsymbol{B}_j$ contains the variables $t_{(j-1)m+1}, \ldots, t_{jm}$. Thus, binding the formula $\boldsymbol{B}_1 \vee \boldsymbol{B}_2 \vee \ldots \vee \boldsymbol{B}_j$ by all the quantifiers $(t_0), \ldots, (t_{jm})$ is meaningful.

LEMMA 1. *The formula*

$$\begin{aligned}(u_1) \ldots (u_n)(A(u_1, \ldots, u_n) \vee B(u_1, \ldots, u_n)) \rightarrow \\ \rightarrow (u_1) \ldots (u_k) \ldots (u_n)(A(u_1, \ldots, u_n) \vee \\ \vee (\exists x_1) \ldots (\exists x_k)B(x_1, \ldots, x_k, u_{k+1}, \ldots, u_n)) \qquad (4)\end{aligned}$$

is a true formula of the predicate calculus.

Proof. We have

$$B(u_1, \ldots, u_n) \rightarrow (\exists x_1) \ldots (\exists x_k)B(x_1, \ldots, x_k, u_{k+1}, \ldots, u_n). \qquad (5)$$

We consider the following true formula in the propositional calculus:

$$(C \rightarrow D) \rightarrow (A \vee C \rightarrow A \vee D).$$

By means of substitutions in this formula, we obtain

$$\begin{aligned}\vdash (B(u_1, \ldots, u_n) \rightarrow \\ \rightarrow (\exists x_1) \ldots (\exists x_k)B(x_1, \ldots, x_k, u_{k+1}, \ldots, u_n)) \rightarrow \\ \rightarrow (A(u_1, \ldots, u_n) \vee B(u_1, \ldots, u_n) \rightarrow A(u_1, \ldots, u_n) \vee \\ \vee (\exists x_1) \ldots (\exists x_k)B(x_1, \ldots, x_k, u_{k+1}, \ldots, u_n)). \qquad (6)\end{aligned}$$

Applying the rule of inference to formulae (5) and (6), we have

$$\vdash A(u_1, \ldots, u_n) \vee B(u_1, \ldots, u_n) \rightarrow A(u_1, \ldots, u_n) \vee \\ \vee (\exists x_1) \ldots (\exists x_k) B(x_1, \ldots, x_k, u_{k+1}, \ldots, u_n). \qquad (7)$$

Further, applying the syllogism rule to the true formula

$$(v_1) \ldots (v_n)(A(v_1, \ldots, v_n) \vee B(v_1, \ldots, v_n)) \rightarrow \\ \rightarrow A(u_1, \ldots, u_n) \vee B(u_1, \ldots, u_n)$$

and to formula (7), we obtain

$$(v_1) \ldots (v_n)(A(v_1, \ldots, v_n) \vee B(v_1, \ldots, v_n)) \rightarrow \\ \rightarrow A(u_1, \ldots, u_n) \vee (\exists x_1) \ldots (\exists x_k) B(x_1, \ldots, x_k, u_{k+1}, \ldots, u_n).$$

Applying the first rule for binding by the quantifier (u_i) successively n times to this formula and renaming the bound variables in it, we obtain formula (4).

We note that if certain of the variables u_i coincide with one another and the number of quantifiers is, by the same token, less than n, then the formula being proved remains true. In this connection, we can assume that the variables $x_1, \ldots, x_k$ which are bound by the existential quantifier do not coincide with one another.

EXAMPLE.

$$(u)(A(u, u) \vee B(u, u)) \rightarrow (u)(A(u, u) \vee (\exists x_1)(\exists x_2) B(x_1, x_2)).$$

The proof for this case remains the same.

LEMMA 2. *The formula*

$$D_j \rightarrow A, \qquad (8)$$

where A is the formula (1), *is deducible for every j.*

Proof. We shall prove this lemma by the method of induction on the index j. The formula

$$(t_0)(t_1) \ldots (t_m) M(t_0, \ldots, t_0, t_1, \ldots, t_m) \rightarrow \\ \rightarrow (\exists x_1) \ldots (\exists x_k)(t_1) \ldots (t_m) M(x_1, \ldots, x_k, t_1, \ldots, t_m)$$

or, if we change the designation of the variables, the formula

$$(t_0)(t_1) \ldots (t_m) M(t_0, \ldots, t_0, t_1, \ldots, t_m) \rightarrow \\ \rightarrow (\exists x_1) \ldots (\exists x_k)(y_1) \ldots (y_m) M(x_1, \ldots, x_k, y_1, \ldots, y_m)$$

is easily deduced. Taking (1) and (2) into consideration, this formula can be written as

$$D_1 \rightarrow A.$$

We shall show that if formula (8) is deducible for $j - 1$, then it is also deducible for j. Let

$$\vdash D_{j-1} \rightarrow A.$$

By virtue of (2), D_j has the form

$$(t_0) \ldots (t_{jm})[C_{j-1} \vee B_j]. \qquad (9)$$

We saw that the variables $t_{(j-1)m+1}, \ldots, t_{jm}$ do not occur in the formula C_{j-1}. In virtue of this, we have

$$(t_{(j-1)m+1}) \ldots (t_{jm})[C_{j-1} \vee B_j] \sim C_{j-1} \vee (t_{(j-1)m+1}) \ldots (t_{jm})B_j \quad (10)$$

since C_{j-1} can be brought out before the quantifiers $(t_{(j-1)m+1}), \ldots, (t_{jm})$. We write formula (9), which is equivalent to D_j, in more detail, taking (10) into consideration:

$$(t_0)(t_1) \ldots (t_{(j-1)m}) [C_{j-1}(t_0, \ldots, t_{(j-1)m}) \vee \\ \vee (t_{(j-1)m+1}) \ldots (t_{jm}) B(t_{i_1}, \ldots, t_{i_k}, t_{(j-1)m+1}, \ldots, t_{jm})],$$

where $t_{i_1}, \ldots, t_{i_k}$ are the variables in the jth set. They are also bound by quantifiers inasmuch as they occur among the variables $t_0, \ldots, t_{(j-1)m}$. Certain of these variables can coincide with one another. We apply formula (4):

$$(u_1) \ldots (u_n)(A(u_1, \ldots, u_n) \vee B(u_1, \ldots, u_n)) \rightarrow \\ \rightarrow (u_1) \ldots (u_k) \ldots (u_n)(A(u_1, \ldots, u_n) \vee \\ \vee (\exists x_1) \ldots (\exists x_k)B(x_1, \ldots, x_k, u_{k+1}, \ldots, u_n)),$$

setting $n = (j-1)m + 1$ and replacing the variables $u_1, \ldots, u_n$ by the variables $t_{i_1}, \ldots, t_{i_k}, \ldots, t_{(j-1)m}$ [these are the same variables $t_0, t_1, \ldots, t_{(j-1)m}$ except that they are written in such an order that the variables of the jth set are written out first]. Moreover, in (4) we replace

$$A(u_1, \ldots, u_n) \text{ by } C_{j-1}(t_{i_1}, \ldots, t_{(j-1)m}),$$

$$B(u_1, \ldots, u_n) \text{ by } (t_{(j-1)m+1}) \ldots (t_{jm})B_j(t_{i_1}, \ldots, t_{i_k}, t_{(j-1)m+1}, \ldots, t_{jm}).$$

This done, we obtain the true formula:

$$(t_{i_1}) \ldots (t_{(j-1)m})[C_{j-1}(t_{i_1}, \ldots, t_{(j-1)m}) \vee \\ \vee (t_{(j-1)m+1}) \ldots (t_{jm})B_j(t_{i_1}, \ldots, t_{i_k}, t_{(j-1)m+1}, \ldots, \\ \ldots, t_{jm})] \rightarrow (t_{i_1}) \ldots, (t_{(j-1)m})[C_{j-1}(t_{i_1}, \ldots, \\ \ldots, t_{(j-1)m}) \vee (\exists x_1) \ldots (\exists x_k)(t_{(j-1)m+1}) \ldots \\ \ldots (t_{jm})B_j(x_1, \ldots, x_k, t_{(-1)m+1}, \ldots, t_{jm})]. \quad (11)$$

The fact that certain of the variables $t_{i_1}, \ldots, t_{i_k}$ can coincide with one another, as we have pointed out, is of no significance.

The left member of implication (11) is equivalent to D_j. In the right member, the second summand of the logical sum has the form

$$(\exists x_1) \ldots (\exists x_k)(t_{(j-1)m+1}) \ldots (t_{jm})B_j(x_1, \ldots, x_k, t_{(j-1)m+1}, \ldots, t_{jm}).$$

Replacing the variables $t_{(j-1)m+1}, \ldots, t_{jm}$ in this formula by $y_1, \ldots, y_m$ respectively and B_j by its value from (2), we obtain the equivalent formula

$$(\exists x_1) \ldots (\exists x_k)(y_1) \ldots (y_m)M(x_1, \ldots, x_k, y_1, \ldots, y_m).$$

And this is nothing but formula A [see (1)]. It follows from this that the right member of implication (11) is equivalent to the formula

$$(t_0) \ldots (t_{(j-1)m})(C_{j-1} \vee A).$$

Since A does not contain the variables $t_0, \ldots, t_{(j-1)m}$, it can be brought out before all the quantifiers. Then the formula

$$A \vee (t_0) \ldots (t_{(j-1)m})C_{j-1}$$

is obtained, which is again one which is equivalent to the right member of implication (11), or, if we replace the second term on the basis of (2),

$$A \vee D_{j-1}. \tag{12}$$

Replacing both members of implication (11) by the equivalent formulae D_j and (12), we arrive at the formula

$$D_j \to D_{j-1} \vee A.$$

But according to our assumption, we have

$$\vdash D_{j-1} \to A.$$

By means of substitutions in the true formula

$$\vdash (A \to B \vee C) \to ((B \to C) \to (A \to C)),$$

we obtain

$$\vdash (D_j \to D_{j-1} \vee A) \to ((D_{j-1} \to A) \to (D_j \to A)).$$

Applying the compound rule of inference, we have

$$\vdash D_j \to A,$$

and the lemma is proved.

Proof of Gödel's theorem. As we have pointed out, to prove Gödel's theorem it is sufficient to restrict ourselves to the consideration of Skolem normal formulae. Let A be an identically true Skolem normal formula. Then A has the form (1). We shall show that in this case at least one of the formulae C_j is deducible in the propositional calculus. Let us assume the contrary, i.e. that none of the formulae C_j is deducible in the propositional calculus. In this case, the formula $\overline{C}_j$—considered as a formula of propositional algebra—will be satisfiable (see §18). In exactly the same way, the formula obtained from $\overline{C}_j$ by an arbitrary replacement of the variables by objects from an arbitrary field will also be satisfiable provided that distinct variables are replaced by distinct objects.

We take an arbitrary countable field of objects which we shall denote by

$$t_1^0, t_2^0, \ldots, t_n^0, \ldots.$$

Let C_j^0 be the formula obtained from C_j by replacing the variables

$$t_{i_1}, \ldots, t_{i_k}, t_{(j-1)m+1}, \ldots, t_{jm}$$

by the objects

$$t_{i_1}^0, \ldots, t_{i_k}^0, t_{(j-1)m+1}^0, \ldots, t_{jm}^0$$

respectively. For arbitrary j, the formula $\overline{C}_j^0$ is a satisfiable formula; and from this it follows that the formula $\overline{B}_1^0 \,\&\, \ldots \,\&\, \overline{B}_j^0$ which is equivalent to it is also satisfiable—as a formula of propositional algebra. In this case, by

Maltsev's theorem, the infinite formula $\prod_{j=1}^{\infty} \bar{B}_j^0$—considered as a formula of propositional algebra—is also deducible. If this formula is satisfiable, then one can give values to the variable propositions occurring in it for which the entire formula has the value T.

We choose definite values for all variables, of the proposition A, occurring in $\prod_{j=1}^{\infty} \bar{B}_j^0$, which transform this formula into T, and henceforth we shall consider only these values of A. In this case all the propositional variables occurring in each of the formulae B_j^0 have a definite value.

Consider the formula B_j^0. According to definition (2), this formula is related to formula A in the following way: B_j^0 represents the expression

$$M(t_{i_1}^0, \ldots, t_{i_k}^0, t_{(j-1)m+1}^0, \ldots, t_{jm}^0),$$

where $(t_{i_1}^0, \ldots, t_{ik}^0)$ is the set of values of the variables $t_{i_1}, \ldots, t_{i_k}$ comprising the jth set.

Since we gave a definite value to every propositional variable occurring in the formula B_j^0, every predicate occurring in M—for example, $A(t_1, \ldots, t_s)$—obtained a definite value at least for certain values of the object variables occurring in it from the field $t_1^0, \ldots, t_n^0, \ldots$. In this distribution of the values for predicates occurring in M there cannot occur any contradiction, i.e. every predicate occurring in M can receive only one value, T or F, for definite values of the variables occurring in it. In fact, although the expression

$$A(t_{p_1}^0, \ldots, t_{p_s}^0)$$

can occur in various B_j^0, in the formula

$$\prod_{j=1}^{\infty} B_j^0$$

it is considered as the same propositional variable and therefore it is replaced by the values T or F in the same way.

Suppose the following predicates occur in the formula A:

$$A_1, A_2, \ldots, A_r.$$

We set into correspondence to each to these predicates A_p an individual predicate defined on the field

$$t_1^0, \ldots, t_n^0, \ldots,$$

which we shall denote by A_p^0 and which is defined in the following way: If for the values $t_{p_1}^0, \ldots, t_{p_s}^0$ of the variables $t_{p_1}, \ldots, t_{p_s}$ occurring in A_p the proposition $A_p(t_{p_1}^0, \ldots, t_{p_s}^0)$ occurs in the formula $\prod_{j=1}^{\infty} \bar{B}_j$, then we assign to $A_p^0(t_{p_1}^0, \ldots, t_{p_s}^0)$ that value which it received when it was considered as a propositional variable in the formula $\prod_{j=1}^{\infty} \bar{B}_j^0$. But if $A_p^0(t_{p_1}^0, \ldots, t_{p_s}^0)$ does not

occur in the formula $\prod_{j=1}^{\infty} \bar{B}_j^0$, we assign to it any value we please. [We recall that for these replacements the formula $\prod_{j=1}^{\infty} \bar{B}_j^0$ receives the value T.]

It is quite easy to see that the formula $\bar{A}$ receives the value T upon replacing the predicates $A_1, \ldots, A_r$ occurring in it by the predicates $A_1^0, \ldots, A_r^0$, i.e. it becomes true in the sense of predicate logic, which was discussed in Chapter III. In fact, the formula $\bar{A}$ is equivalent to the following formula:

$$(x_1) \ldots (x_k)(\exists y_1) \ldots (\exists y_m)\bar{M}(x_1, \ldots, x_k, y_1, \ldots, y_m).$$

Consider now an arbitrary set of values of the variables $x_1, \ldots, x_k$ taken from the field $t_1^0, \ldots, t_i^0, \ldots$. Every such set of values represents one of the sets

$$(t_{i_1}^0, \ldots, t_{i_k}^0).$$

Let the index of this set be j. The formula

$$\bar{M}(t_{i_1}^0, \ldots, t_{i_k}^0, t_{(j-1)m+1}^0, \ldots, t_{jm}^0)$$

represents the formula

$$\bar{B}_j(t_{i_1}^0, \ldots, t_{i_k}^0, t_{(j-1)m+1}^0, \ldots, t_{jm}^0).$$

This formula receives the value T upon replacement of the predicates $A_1, \ldots, A_r$ occurring in it by the predicates $A_1^0, \ldots, A_r^0$ [inasmuch as the entire formula $\prod_{j=1}^{\infty} \bar{B}_j^0$ takes on the value T for such a replacement]. This means that for an arbitrary set of values of the variables $x_1, \ldots, x_k$ on the field $t_1^0, \ldots, t_p^0, \ldots$ there exists a set of values of the variables $y_1, \ldots, y_m$—namely, $t_{(j-1)m+1}^0, \ldots, t_{jm}^0$—such that the formula

$$\bar{M}(t_{i_1}^0, \ldots, t_{i_k}^0, t_{(j-1)m+1}^0, \ldots, t_{jm}^0)$$

receives the value T. But then the formula

$$(x_1) \ldots (x_k)(\exists y_1) \ldots (\exists y_m)\bar{M}(x_1, \ldots, x_k, y_1, \ldots, y_m)$$

itself receives the value T upon replacement of the predicates $A_1, \ldots, A_r$ by the predicates $A_1^0, \ldots, A_r^0$ for the field $t_1^0, \ldots, t_n^0, \ldots$. Consequently, this formula is *satisfiable* on the field $t_1^0, \ldots, t_n^0, \ldots$. However, it is equivalent to the negation of the formula

$$(\exists x_1) \ldots (\exists x_k)(y_1) \ldots (y_m)M(x_1, \ldots, x_k, y_1, \ldots, y_m),$$

which is the formula A and which, by assumption, is identically true. We have arrived at a contradiction inasmuch as the negation of an identically true formula cannot be satisfiable on any field. It follows from the contradiction just obtained that the formula $\prod_{j=1}^{\infty} \bar{B}_j$ cannot be satisfiable.

In this case, our assumption that none of the formulae C_j is deducible in the propositional calculus is false and, consequently, there exists a formula

C_{j_0} which is deducible in the propositional calculus and consequently in the predicate calculus also. In this case, the formula D_{j_0} which represents

$$(t_0) \ldots (t_{j_0 m})C_{j_0}$$

is also deducible in the predicate calculus (by means of successive applications of the derived rule for binding by a quantifier). But we know that every formula $D_j \to A$ is deducible in the predicate calculus; therefore the formula A is also deducible in the predicate calculus, which is what we required to prove.

§20. Systems of axioms in the predicate calculus

The formal meaning of formulae which are deducible in the predicate calculus is sufficiently fully established in Chapters III and IV. These formulae represent identically true propositions—in other words, they are logical tautologies. Conversely, every identically true formula is deducible in the predicate calculus (see §19).

From what was stated above, it is clear that it is impossible to introduce, in the predicate calculus, any propositions which are to any degree formal in essence, in particular mathematical. However, if to the axioms of the predicate calculus there are adjoined any non-deducible formulae in the role of new axioms (retaining the same rules of inference), then another calculus is obtained in which other formulae—besides the identically true formulae—are deducible.

Although the addition to the axioms of the predicate calculus of new axioms of the type we have considered up to this point, extends the content of this calculus, this extension is not very significant. An example of such an extension of the system of axioms of the predicate calculus is, for example, the addition of the formula considered above:

$$(\exists x)F(x) \to (x)F(x),$$

which is not deducible in the predicate calculus. The formal meaning of this axiom consists in the affirmation that all objects are identical.

In the informal exposition of predicate logic we also considered systems of axioms. But there we considered various types of symbols for objects and predicates. Namely, objects as well as predicates were separated into variables and individuals. In the predicate calculus, up to this point we have dealt only with object and predicate variables. In order to extend the domain of application of the predicate calculus and to obtain the possibility of describing the present mathematical disciplines by means of axioms it is necessary to introduce into the predicate calculus also individual objects and individual predicates.

We shall not introduce any special symbols for individual objects. We shall use lower case Latin letters for them, stating in every case which letters we shall call individual objects. Unless specifically stated otherwise,

we shall assume that we are dealing with object variables. We shall most frequently use the letters at the beginning of the alphabet to denote individual objects. In certain cases we shall denote individual objects by other symbols —for instance, by numerals. In exactly the same way, we shall also introduce individual predicates into the formalism. We shall use the same symbols for them as for predicate variables with the corresponding reservations, or we shall depict them by means of symbols having a universally accepted form. Thus, for example, we shall depict the individual predicate of two variables, called the "predicate of equality", by the symbol $=$. Another individual predicate, called the "predicate of order", will be denoted by the symbol $<$.

The concept of an elementary formula will also be somewhat extended. To the elementary formulae we shall add expressions of the form

$$F(x_1, \ldots, x_n),$$

where F is a predicate variable or individual predicate of n variables, and the letters $x_1, \ldots, x_n$ can be object variables as well as individual objects. Elementary formulae of the form $F(x_1, \ldots, x_n)$—if they contain object variables—will also be called predicate variables or individual predicates respectively depending on what the predicate F is. For individual predicates we shall sometimes write these elementary formulae differently. Thus, the predicate of equality will be written down in the form

$$x = y \text{ (and not } = (x, y)),$$

and the predicate of order in the form

$$x < y.$$

The remainder of the definition of a formula remains unchanged. It must only be noted that the operation of binding by a quantifier does not extend to individual objects.

The rule for substitution in a predicate variable requires a somewhat more precise formulation.

The substitution

$$R^{B}_{F(t_1, \ldots, t_n)}(A),$$

where F is a predicate variable of n arguments, represents the replacement of every expression of the form $F(x_1, \ldots, x_n)$ occurring in the formula A by the formula $B(x_1, \ldots, x_n)$, where here the letters $x_1, \ldots, x_n$ can be object variables as well as individual objects.

The expressions $(a)A(a)$ and $(\exists a)A(a)$, where a is an individual object, are not formulae.

All axioms of the predicate calculus and all rules of inference—except the rule for substitution in free object variables—remain the same as before. The rule for substitution in object variables is extended and is formulated in the following manner:

If A is a true formula, then the formula A' obtained from A by the replace-

ment of an arbitrary free object variable by another object variable or by an individual object everywhere where this variable occurs in A is also true.

We retain the nomenclature *predicate calculus* for the system thus obtained. It is easily seen that the additions introduced into the predicate calculus do not change its essence. The axioms of the predicate calculus do not contain individual objects and predicates. If we deduce a formula in which we distinguish individual and variable objects and predicates, then this deduction would still be legitimate under the hypothesis that all the objects occurring in the formula (except, of course, the bound object variables) and all the predicates are variable. This assertion is true because the only modifications in the rules of inference consist in the restriction of their application to individual objects and predicates. Thus, the additions introduced contribute nothing to the predicate calculus itself. They, however, turn out to be essential for the construction of abstract logical calculi which are intended for the description not of logical but of mathematical disciplines. Let

$$A_1, A_2, \ldots, A_n$$

be formulae of the predicate calculus which can now contain variable as well as individual predicates and objects. We adjoin these formulae to the predicate calculus, retaining the previous rules of inference. We then obtain a formal system in which all true formulae of the predicate calculus are deducible but also other formulae are deducible which are not deducible in the predicate calculus. This situation always holds if at least one of the formulae A_i is not deducible in the predicate calculus.

As an example, we consider the system of equality axioms which we have already encountered in Chapter III. We shall assume that the following axioms are adjoined to the axioms of the predicate calculus:

1. $x = x$,
2. $x = y \rightarrow (A(x) \rightarrow A(y))$.

In these axioms, $x = y$ is an *individual* predicate and $A(x)$ is a predicate *variable*. It is easily seen that none of these axioms can be deduced in the predicate calculus. In fact, formulae which are deducible in the predicate calculus do not contain individual predicates. Therefore, if the formula $x = x$ were deduced in the predicate calculus, then the predicate $x = x$ could be considered as a predicate variable. But the provability of an elementary predicate variable is impossible inasmuch as it leads directly to a contradiction. Substituting the formula $\overline{x = x}$ for $x = x$ we would also obtain a true formula in the predicate calculus and we would by the same token obtain a contradiction in the predicate calculus itself. The non-deducibility of the second formula in the predicate calculus can also be shown in an analogous manner. We shall show further that it is possible to deduce the fundamental properties of equality from axioms 1 and 2.

By means of adjunction to the axioms of the predicate calculus of these or other axioms we can describe various mathematical disciplines. Arithmetic, the theory of real numbers, geometry and, finally, the theory of sets can be described in this way within very wide limits. In general, an arbitrary deductive system can be expressed by the means pointed out above. Thus, there is essentially no need for the construction of other formalisms.

Besides, the consideration of an arbitrary system of axioms raises the problem of its intrinsic consistency. This problem, as we saw, is vital inasmuch as every intrinsically inconsistent system of the type under consideration is distinguished by the property that in it all formulae are simultaneously true and false. It is therefore unsuitable for the pursuit of knowledge. The question of consistency is, however, connected with special, not completely removable, difficulties. The situation is this that if we carry out any proof of the consistency of a definite calculus then this line of reasoning itself must be based on assumptions which themselves cannot be deduced from the axioms of the given system. (Every line of reasoning—and this includes an arbitrary line of reasoning about the consistency of any system—can be formalized and presented in the form of a deduction from axioms which are expressed by formulae of the predicate calculus.)

So, in order to prove the consistency of any calculus, it is necessary to make use of assumptions (antecedents or premisses) which are not deducible from the axioms of the given calculus. Thus, for the solution of the problem of consistency it turns out to be necessary for it to be possible to draw from some source stronger and stronger premisses of such a type that we already have complete confidence in the consistency of them themselves. One would have thought that that finitary system of reasoning, in the framework of which we define logical systems and carry out reasonings about them, is such a premiss. But it turns out that Hilbert's finitism does not suffice for the proof of the consistency of arithmetic. In practice, the source of premisses made use of at the present time for the solution of consistency problems of sufficiently powerful mathematical theories remains the theory of sets, and the proof of the consistency of any system is caried out in the following manner: we prove that if the theory of sets in one or other formulation is consistent then the given system of interest to us is also consistent.

It is again possible to follow such a line of reasoning in the framework of Hilbert's finitism as both the theory of sets to the extent we need it and the system under investigation can already be described axiomatically and even by means of the predicate calculus. Thus, instead of the problem of the consistency of a given system, we solve a much weaker problem—the reduction of the problem of the consistency of the system under consideration to the problem of the consistency of some formal set-theoretic system.

The possibility exists of finding principles allowing one to form premisses for the proof of the consistency of powerful formal systems and perhaps even

of set-theoretic systems. Moreover, these principles are such that belief in the consistency of the premisses themselves is very much more well founded than is the case for set-theoretic systems. However, this goes beyond the bounds of the present book and so we shall not touch upon these problems here. In the sequel, we shall restrict ourselves to proofs of consistency only within the bounds allowed by Hilbert's finitism.

Besides the problem of the consistency of a system of axioms, there always arises the problem of the independence of axioms, i.e. of the non-deducibility of any axiom from the remaining axioms. In the problem of the independence of axioms, there are no new fundamental difficulties. In general, the problem of the independence of axioms is usually posed for consistent systems or for systems whose consistency is assumed. In the latter case, in the problem of the independence of axioms one restricts oneself to the reduction of this problem to that of the consistency of the given system.

As we have said, although we have no fundamental necessity in the description of mathematical disciplines to introduce any modification in the symbols, formulae and rules of inference of the predicate calculus, none the less, for the sake of convenience, it is reasonable in certain cases to resort to various modifications in the indicated direction.

In the sequel, for the description of arithmetic we shall extend the set of svmbols of the predicate calculus by introducing further new symbols. The introduction of new symbols entails a generalization of the concept of formula and requires a corresponding change in the formulations of the rules of inference.

CHAPTER V

AXIOMATIC ARITHMETIC

§1. Terms. The extended predicate calculus

In this chapter we shall give a description of arithmetic in the form of an axiomatic system. As we have already indicated at the end of the preceding chapter, we could obtain such a description by adjoining to the predicate calculus (comprising individual predicates and objects) certain new axioms. However, such a discussion would be cumbersome and inconvenient. Therefore, before writing down the axioms of arithmetic, we extend predicate calculus further by introducing new symbols into it. These symbols are lower-case Greek letters:

$$\alpha, \beta, \ldots, \phi, \psi, \ldots.$$

Together with the object variables and the individual objects, they form what we shall call *object functions* and *object constants*. Strings of the following form correspond to them:

$$\alpha(x), \beta(x, y), \psi(x_1, x_2, \ldots, x_n), \eta(a), \xi(a, b), \ldots. \tag{1}$$

These expressions are constructed in exactly the same way as predicates [for example, $F(x, y, z)$], but with the difference that lower-case Greek letters occur in them instead of upper-case Latin letters (in our example F).

The concepts of object function and object constant are defined completely in the following way.

1. Expressions of the form (1) are object functions. We shall call these expressions *elementary object functions*.

2. The result of replacing object variables in an object function by object functions is also an object function.

3. Individual objects are object constants.

4. The result of replacing some, but not all, variables in an object function by individual objects is an object function.

5. The result of replacing all object variables in an object function by individual objects is an object constant.

Examples of object functions and object constants are:

$$\alpha(\beta(x)), \beta(\alpha(x), \gamma(a)), \beta(a, \gamma(a)).$$

We combine the concepts of object variable, object function and object constant into the one concept *term*. By introducing terms into the predicate

calculus, we also extend the rules for the formation of formulae. Namely, to the elementary formulae, we adjoin the rows obtained by replacing an arbitrary object variable in an elementary predicate by an arbitrary term.

Thus, for example,

$$A(\psi(x)),\ B(x,\ \phi(y,\ x)),\ F(a,\ x,\ \psi(a,\ y)),\ G(a,\ \psi(a),\ \phi(a,\ b))$$

are elementary formulae; otherwise the rules for the formation of formulae do not change.

We note that the terms themselves are not formulae. This corresponds to the informal meaning of these concepts: a term is an object and a formula is a proposition about objects. Similar to the way we used metalogical notation in the form of upper-case bold letters for formulae, we shall denote terms in the form of lower-case bold letters, for example, $\boldsymbol{a}$, $\boldsymbol{b}$, ..., or, if we wish to express the fact that the term contains the variables $x_1, \ldots, x_n$,

$$\boldsymbol{a}(x_1, \ldots, x_n).$$

Further, we extend the rules of deduction by modifying the *substitution rules*. We shall now formulate the rule of substitution in an object variable in the following way:

If, in the valid formula $\boldsymbol{A}$, we replace the arbitrary free object variable by a term which does not contain variables which are bound in the formula $\boldsymbol{A}$, then the formula obtained after this replacement will also be valid.

The rule of substitution in a propositional variable remains the same as in the predicate calculus.

The rule of substitution in a predicate variable changes somewhat.

The substitution, expressed by the symbol

$$R^{\boldsymbol{B}(t_1, \ldots, t_n)}_{F(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)}(\boldsymbol{A})$$

where F is a predicate of n variables, is the replacement of every expression of the form $F(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ in the formula $\boldsymbol{A}$, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are certain terms, by the formula $\boldsymbol{B}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$, obtained from $\boldsymbol{B}(t_1, \ldots, t_n)$ by the replacement of the variables $t_1, \ldots, t_n$ by the variables $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, respectively (cf. §4, Chapter IV).

Every calculus obtained from the predicate calculus by adjoining certain object functions and the following equality axioms:

$$\text{VI.1. } x = x,$$

$$\text{VI.2. } x = y \rightarrow (A(x) \rightarrow A(y))$$

and extending the rules of substitution as indicated above will be called an *extended predicate calculus*. Obviously, all formulae which we derived in the predicate calculus are derivable in an extended predicate calculus as well as in any system obtained by adjoining to the extended predicate calculus any axioms and new rules for the formation of valid formulae we please. The truth of this is clear, since all axioms and rules of deduction of the predicate

calculus, on the basis of which the derived formulae are deduced, are retained in all extensions.

The expression $\vdash A$ which denotes the deducibility of the formula A in the predicate calculus will now be used also for formulae which are deducible in the extended predicate calculus and in other formalisms.

Using this symbol for various formalisms will not lead to confusion as it will always be clear from the context in which formalism the given formula is deduced.

§2. Properties of the equality predicate and of object functions

We shall derive the fundamental properties of the equality predicate.

(1) $$\vdash x = y \rightarrow y = x. \tag{1}$$

In axiom VI.2 we perform a substitution consisting in replacing $A(t)$ by the formula $t = x$. We then obtain

$$\vdash x = y \rightarrow (x = x \rightarrow y = x).$$

Applying the rule of interchanging antecedents, we obtain

$$\vdash x = x \rightarrow (x = y \rightarrow y = x).$$

Next, applying the rule of conclusion, we obtain the required formula:

$$x = y \rightarrow y = x.$$

from axiom VI.1.

(2) The following property of equality is called transitivity:

$$\vdash x = y \rightarrow (y = z \rightarrow x = z). \tag{2}$$

We shall first deduce the formula

$$x = y \rightarrow (A(y) \rightarrow A(x)).$$

Relabelling the variables in axiom VI.2, we obtain

$$\vdash y = x \rightarrow (A(y) \rightarrow A(x)). \tag{3}$$

From this formula and the formula (1) which we deduced above, applying the syllogism rule, we obtain the required formula:

$$\vdash x = y \rightarrow (A(y) \rightarrow A(x)). \tag{4}$$

Substituting in (4) the formula $t = z$ in place of the predicate $A(t)$, we obtain (2).

COROLLARY 1. $\vdash x = y \rightarrow (A(x) \sim A(y))$.

Combining (4) with axiom VI.2, we obtain

$$\vdash x = y \rightarrow (A(x) \rightarrow A(y)) \,\&\, (A(y) \rightarrow A(x)),$$

or

$$\vdash x = y \rightarrow (A(x) \sim A(y)),$$

which is what we required to prove.

We shall say that terms $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are equal in a calculus containing the equality axiom if the formula*

$$\boldsymbol{a} = \boldsymbol{b}$$

is deducible in this calculus. In this case, the following assertion holds:

If the terms $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are equal in some calculus, then the formula*

$$A(\boldsymbol{a}) \sim A(\boldsymbol{b})$$

is deducible in it, where $A(\boldsymbol{a})$ and $A(\boldsymbol{b})$ are obtained by the replacement of an arbitrary variable x occurring in the formula A.

In fact, performing a substitution in the formula

$$x = y \rightarrow (A(x) \sim A(y)),$$

we have, first, that

$$\vdash x = y \rightarrow (A(x) \sim A(y)),$$

and, next, that

$$\vdash \boldsymbol{a} = \boldsymbol{b} \rightarrow (A(\boldsymbol{a}) \sim A(\boldsymbol{b})),$$

since $\vdash \boldsymbol{a} = \boldsymbol{b}$ holds, and, applying the rule of implication, we obtain

$$\vdash A(\boldsymbol{a}) \sim A(\boldsymbol{b}).$$

From this it follows that if the formula $A(\boldsymbol{a})$ is deducible in a calculus containing the equality axiom, then the formula $A(\boldsymbol{b})$, obtained by replacing the term $\boldsymbol{a}$ by the term $\boldsymbol{b}$, equal to it, is also deducible in this calculus.

We note that in replacing a term which occurs in a formula A by an equal term, it is not necessary to replace it where ever it occurs in the formula A. Suppose the term $\boldsymbol{a}$ occurs in various parts of the formula A; we represent the formula A in the form $A(\boldsymbol{a}, \boldsymbol{a})$. If $\boldsymbol{b}$ equals $\boldsymbol{a}$, then

$$\vdash A(\boldsymbol{a}, \boldsymbol{a}) \sim A(\boldsymbol{a}, \boldsymbol{b})$$

holds.

In fact, let us take variables x and y which do not occur in the terms $\boldsymbol{a}$ and $\boldsymbol{b}$; then, replacing the predicate $A(t)$ by the predicate $A(\boldsymbol{a}, t)$ in the formula

$$x = y \rightarrow (A(x) \sim A(y)),$$

we have

$$\vdash x = y \rightarrow (A(\boldsymbol{a}, x) \sim A(\boldsymbol{a}, y)).$$

Substituting $\boldsymbol{a}$ for x and $\boldsymbol{b}$ for y, we obtain

$$\vdash \boldsymbol{a} = \boldsymbol{b} \rightarrow (A(\boldsymbol{a}, \boldsymbol{a}) \sim A(\boldsymbol{a}, \boldsymbol{b})).$$

From this and the equality $\boldsymbol{a} = \boldsymbol{b}$, it follows that

$$\vdash A(\boldsymbol{a}, \boldsymbol{a}) \sim A(\boldsymbol{a}, \boldsymbol{b}),$$

so that, replacing $\boldsymbol{a}$ in only one place by $\boldsymbol{b}$, we obtain an equivalent formula.

We shall derive some valid formulae which connect the equality predicate with object functions:

$$\vdash (\exists x)[\boldsymbol{a}(x) = \boldsymbol{a}(y)]. \tag{5}$$

From the axiom $F(z) \to (\exists x)F(x)$, we obtain, by a substitution,

$$\vdash \boldsymbol{a}(z) = \boldsymbol{a}(y) \to (\exists x)[\boldsymbol{a}(x) = \boldsymbol{a}(y)].$$

Replacing the free variable z by y, we obtain

$$\vdash \boldsymbol{a}(y) = \boldsymbol{a}(y) \to (\exists x)[\boldsymbol{a}(x) = \boldsymbol{a}(y)].$$

Applying the rule of deduction, we obtain (5).

$$\vdash (x)A(x) \to (x)(y)A(\boldsymbol{a}(x, y)). \tag{6}$$

From the axiom $(x)A(x) \to A(z)$, replacing the free term z in it by the term $\boldsymbol{a}(u, v)$, we obtain

$$\vdash (x)A(x) \to A(\boldsymbol{a}(u, v)).$$

Applying twice the first rule of binding with a quantifier and relabelling the bound variables, we obtain formula (6). Clearly, an analogous valid formula can be obtained for the function $\boldsymbol{a}$ with an arbitrary number of variables.

$$\vdash x = y \to (\boldsymbol{a}(x) = \boldsymbol{a}(y)). \tag{7}$$

Replacing $A(t)$ by $\boldsymbol{a}(x) = \boldsymbol{a}(t)$ in the second equality axiom, we obtain

$$\vdash x = y \to [\boldsymbol{a}(x) = \boldsymbol{a}(x) \to \boldsymbol{a}(x) = \boldsymbol{a}(y)].$$

And, interchanging the antecedents, we obtain

$$\vdash \boldsymbol{a}(x) = \boldsymbol{a}(x) \to [x = y \to \boldsymbol{a}(x) = \boldsymbol{a}(y)].$$

Finally, eliminating the true antecedent, we obtain formula (7).

We shall now prove the following formula:

$$(x = u) \& (y = v) \to [g(x, y) = g(u, v)]. \tag{8}$$

We derive first the two formulae:

$$x = u \to [g(x, y) = g(u, y)], \tag{9}$$

$$y = v \to [g(u, y) = g(u, v)]. \tag{10}$$

From the valid formula

$$x = u \to (u = y \to x = y),$$

by replacing the free variables by a term, we obtain

$$g(x, y) = g(u, y) \to \{g(u, y) = g(u, v) \to g(x, y) = g(u, v)\}.$$

Applying the syllogism rule to formula (9) and the last formula, we obtain

$$x = u \to \{g(u, y) = g(u, v) \to g(x, y) = g(u, v)\}.$$

Interchanging the antecedents in this formula, we obtain the formula

$$g(u, y) = g(u, v) \to \{x = u \to g(x, y) = g(u, v)\}. \tag{11}$$

Applying the syllogism rule to formulae (10) and (11), we obtain

$$y = v \to \{x = u \to g(x, y) = g(u, v)\}.$$

Combining the antecedents in the product and interchanging the factors, we obtain the formula (8).

The informal meaning of these theorems is that the object functions we have introduced are single-valued, i.e. equal values of a function correspond to equal values of the arguments.

The fact that every object function $r(u_1, \ldots, u_n)$ satisfies

$$u_1 = v_1 \mathbin{\&} u_2 = v_2 \mathbin{\&} \ldots \mathbin{\&} u_n = v_n \rightarrow (r(u_1, \ldots, u_n) = r(v_1, \ldots, v_n)),$$

will be expressed by the phrase "*object functions are single-valued*".

§3. Equivalence relations

In mathematics, we continually have to deal with relations which express some similarity between the objects under consideration—for example, the equality of numbers, the similarity of geometric figures, the logical equivalence of propositions, etc. We shall call these relations *equivalence relations*, utilizing here the term "equivalence" in another—wider—sense than the term "equivalence of formulae" which we have used constantly in a specially well-defined logical sense.

One can characterize an equivalence relation by the following properties:

(1) *Reflexive:* every object is equivalent to itself.

(2) *Symmetrical:* if x is equivalent to y, then y is equivalent to x.

(3) *Transitive:* if x is equivalent to y and y is equivalent to z, then x is equivalent to z.

If an equivalence relation $S(x, y)$ is established for any field of objects M, then S divides the field M into mutually exclusive classes such that all the elements in any one class are equivalent and any two equivalent elements belong to the same class.

EXAMPLE. Let M be the set of all integers and suppose $S(x, y)$ is congruence modulo 3:

$$x \equiv y \pmod 3.$$

In this case M is partitioned into three classes. The first class consists of integers of the form $3n$, which are divisible by 3. The second of integers of the form $3n + 1$, which yield a remainder of 1 upon division by 3. And the third class consists of integers of the form $3n + 2$ which have a remainder of 2 upon division by 3.

Equivalence relations can be expressed by means of formulae of the predicate calculus. To this end it is necessary to write the equivalence properties in the form of axioms. We shall denote the equivalence relation by the symbol $\approx$. Then the equivalence properties can be written as follows:

1. $x \approx x$,
2. $x \approx y \rightarrow y \approx x$,
3. $x \approx y \rightarrow [y \approx z \rightarrow x \approx z]$.

The equality relation possesses these same properties; the property of being reflexive coincides with the first axiom of equality, and the remaining two—symmetry and transitivity—as we saw, are consequences of the axioms of equality in the extended predicate calculus.

In informal mathematics and in the naïve theory of sets there exists one special form of equivalence relation, namely *identity*. If x and y are identical, then they denote one and the same object. Clearly, in this case each of the classes into which M decomposes consists of only one object. If one considers the predicate calculus from the viewpoint of naïve set theory, then equality, which is defined by axioms VI.1 and VI.2, is in fact set-theoretic identity. We stress here once again that the second axiom of equality has the following informal meaning: "if x and y are identical, then everything that can be said about x can be said about y also".

It can also be formally proved that if two objects are identical, then they are equivalent. Speaking more precisely, one can deduce the formula

$$x = y \rightarrow x \approx y$$

from the axioms of equality and equivalence. In fact, substituting $x \approx t$ in place of $A(t)$ in the equality axiom VI.2, we obtain the formula

$$x = y \rightarrow [x \approx x \rightarrow x \approx y].$$

And, interchanging the antecedents:

$$x \approx x \rightarrow [x = y \rightarrow x \approx y].$$

We now obtain the required formula by deleting the true formula $x \approx x$.

In the informal treatment of the system of axioms of the predicate calculus we assumed that the predicate variables appearing in the axioms could be replaced by arbitrary predicates defined on a given field. We can change the character of the interpretation of the system of axioms by requiring that the given system of axioms be true for a given field under replacements of the predicate variables, not by arbitrary predicates defined on the given field, but only by those which belong to some definite set forming a subset of the set of all predicates. In this case, for some subset of predicates, axioms VI.1 and VI.2 define an equivalence relation which can be called a relative equality or an equality relative to the given system of predicates.

§4. Deduction theorem

The deduction theorem proved in the preceding chapter also generalizes to the extended predicate calculus. In order to formulate this theorem, it is first necessary, as was the case before, to define the concept of the deducibility of a formula $\boldsymbol{B}$ from a formula $\boldsymbol{A}$.

1. All true formulae of the extended predicate calculus, which do not lead to collision of variables with $\boldsymbol{A}$, are deducible from $\boldsymbol{A}$.

2. The formula $\boldsymbol{A}$ is deducible from $\boldsymbol{A}$.

3. If $\boldsymbol{B}_1$ and $\boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2$ are deducible from $\boldsymbol{A}$, then $\boldsymbol{B}_2$ is deducible from $\boldsymbol{A}$.

4. If $\boldsymbol{B}_1 \rightarrow \boldsymbol{B}_2(x)$ is deducible from $\boldsymbol{A}$ and the variable x is contained neither in $\boldsymbol{A}$ nor in $\boldsymbol{B}_1$, then the formula $\boldsymbol{B}_1 \rightarrow (x)\boldsymbol{B}_2(x)$ is also deducible from $\boldsymbol{A}$.

5. If the formula $\boldsymbol{B}_1(x) \rightarrow \boldsymbol{B}_2$ is deducible from $\boldsymbol{A}$ and x is contained neither in $\boldsymbol{B}_2$ nor in $\boldsymbol{A}$, then the formula $(\exists x)\boldsymbol{B}_1(x) \rightarrow \boldsymbol{B}_2$ is also deducible from $\boldsymbol{A}$.

6. If the formula $\boldsymbol{B}$ is deducible from $\boldsymbol{A}$, then the formula $\boldsymbol{B}'$, which is obtained from $\boldsymbol{B}$ by a substitution in a propositional variable or predicate variable, not occurring in the formula $\boldsymbol{A}$, such that $\boldsymbol{B}'$ does not come into collision with the variables of $\boldsymbol{A}$, is also deducible from the formula $\boldsymbol{A}$.

7. If the formula $\boldsymbol{B}$ is deducible from the formula $\boldsymbol{A}$ and the formula $\boldsymbol{B}'$ is obtained from $\boldsymbol{B}$ by a substitution in a free object variable, which does not occur in $\boldsymbol{A}$, such that $\boldsymbol{B}'$ does not lead to a collision of variables with $\boldsymbol{A}$, then the formula $\boldsymbol{B}'$ is deducible from $\boldsymbol{A}$.

8. If the formula $\boldsymbol{B}$ is deducible from the formula $\boldsymbol{A}$ and $\boldsymbol{B}'$ is a formula obtained from $\boldsymbol{B}$ by means of renaming the bound variables such that $\boldsymbol{B}'$ does not lead to a collision of variables with $\boldsymbol{A}$, then the formula $\boldsymbol{B}'$ is deducible from $\boldsymbol{A}$.

Thus, the definition of deducibility in the extended predicate calculus coincides precisely with the corresponding definition for the predicate calculus. The content of this definition, however, is somewhat different inasmuch as the rules for substitution in the extended predicate calculus differ from the corresponding rules in the predicate calculus. The formulation of the deduction theorem for the extended predicate calculus remains the same as above and its proof is carried out analogously.

DEDUCTION THEOREM. *If the formula $\boldsymbol{B}$ is deducible from the formula $\boldsymbol{A}$ in the indicated sense, then the formula $\boldsymbol{A} \rightarrow \boldsymbol{B}$ is deducible in the extended predicate calculus.*

§5. Axioms of arithmetic

We introduce into our system the individual object 0 and the object function $\lambda(x)$. In this case, the following functions and constants will also appear in our system:

$$\lambda(\lambda(x)),\ \lambda(\lambda(\lambda(x))),\ \ldots,\ \lambda(\lambda(\ldots(x)\ldots)),\ \lambda \text{ occurring } n \text{ times},\ \ldots,$$
$$\lambda(0),\ \lambda(\lambda(0)),\ \ldots,\ \lambda(\lambda(\ldots(0)\ldots)).$$

For the sake of brevity, we shall write the function $\lambda(x)$ in the form x'. The operation of replacement of an argument, for instance,

$$\lambda(\boldsymbol{a}(x, y)),$$

will be written in this symbolism as

$$(\boldsymbol{a}(x, y))'.$$

Then the expressions

$$\lambda(x), \lambda(\lambda(x)), \ldots, \lambda(\lambda(\ldots(\lambda(x))\ldots)), \ldots$$

will be depicted in the form

$$x', (x')', ((x')')', \text{ and so forth.}$$

We shall simplify the writing of these expressions still further. Namely, we shall write $(x')'$ in the form x'', $((x')')'$ in the form x''', and so on.

The object variables

$$\lambda(0), \lambda(\lambda(0)), \ldots$$

are now depicted in the form

$$0', 0'', \ldots, 0'\overset{n \text{ times}}{\cdots}{}', \ldots.$$

These constants, together with the object 0, *we shall call numerals.* The numeral represented by the symbol 0 with n primes will be denoted by $0^{(n)}$.

In the informal discussion of the predicate calculus (Chapter III) we have already considered the axioms of arithmetic, or, more precisely speaking, the axioms of the order relations of the sequence of natural numbers (the axioms of the order relations of arithmetic). We shall now express these same properties of the natural number sequence by means of formulae of some calculus. To this end, to the extended predicate calculus (see page 204), containing the object 0 and the function x', the term "less than" we add the following axioms

VII. Order Axioms

1. $\overline{x < x}$,
2. $x < y \rightarrow (y < z \rightarrow x < z)$,
3. $x < x'$.

VIII. Axiom of Complete Induction: $A(0) \,\&\, (x)(A(x) \rightarrow A(x')) \rightarrow A(y)$.

The rules of inference in the new calculus are the same as those in the extended predicate calculus. This system of axioms as far as its informal meaning is concerned expresses the same thing as the system of axioms of the natural number sequence, which we considered in §8, Chapter III. Speaking more precisely, if we consider our logical system from the informal viewpoint, discussed in Chapter III, then the concepts under consideration—objects, predicates and logical formulae—are meaningful in relation to the given set or field. The object function $g(x_1, \ldots, x_n)$ represents a function in the usual sense of the word, and is defined on the given field and takes on values from the given field.

From this point of view we can show that axioms VI, VII, VIII are interpretable by means of an arbitrary ordered set which is similar to the sequence of natural numbers in which for the object 0 we take the element preceding (in the sense of the order relation of the given ordered set) all the remaining

elements, and the value of the function x' is the element which follows immediately after the element x. And, conversely, only a set which is similar to the sequence of natural numbers satisfies the given system of axioms. Thus, if one treats the axioms formally, the system of axioms VI, VII, VIII is equivalent to the system which we have already considered in §8, Chapter III. It is consequently interpretable and complete to within isomorphism. However, we are not now considering the system of axioms VI, VII, VIII in the spirit of Chapter III. These axioms, together with the axioms and rules of inference of the predicate calculus, constitutes a formalism which is defined on the basis of the principles of finitism. Therefore the informal treatment of these axioms introduced above can have only heuristic significance for the solution of problems arising in the study of the given formalism. Axioms VI, VII, VIII do not exhaust formal arithmetic—they do not contain a description of arithmetic operations. We shall extend this calculus still further in the sequel. But we shall first consider the system of axioms VI, VII, VIII in more detail.

§6. Examples of deducible formulae

We introduce—in the form of examples—the formal proofs of some propositions which are deducible from axioms VI, VII, VIII.

THEOREM 1. $\vdash 0 < x'$.

Proof. Replacing in the complete induction axiom the predicate variable $A(x)$ by the predicate $0 < x'$, we obtain

$$\vdash 0 < 0' \mathbin{\&} (x)\,[0 < x' \to 0 < x''] \to 0 < y'. \tag{1}$$

By means of substitution in axiom VII.3, we conclude that

$$\vdash 0 < 0'$$

and

$$\vdash x' < x''.$$

By substitutions in axiom VII.2, we obtain

$$\vdash 0 < x' \to (x' < x'' \to 0 < x'').$$

Interchanging the antecedents, we obtain

$$\vdash x' < x'' \to (0 < x' \to 0 < x'').$$

Since $x' < x''$ is a deducible formula, the rule of inference gives

$$\vdash 0 < x' \to 0 < x''.$$

And, finally, by the product rule for binding by a quantifier, we obtain

$$\vdash (x)\,(0 < x' \to 0 < x'').$$

Applying the rule

$$\frac{A,\ B}{A \mathbin{\&} B}$$

to the formulae $0 < 0'$ and $(x)\,(0 < x' \to 0 < x'')$, we have

$$\vdash 0 < 0' \,\&\, (x)\,(0 < x' \to 0 < x''). \tag{2}$$

Applying the inference rule to (2) and (1), we obtain

$$\vdash 0 < y'$$

and consequently

$$\vdash 0 < x'.$$

THEOREM 2. $\vdash \overline{x < 0} \to x = 0 \vee 0 < x.$

Proof. Replacing the predicate variable $A(x)$ in axiom VIII by the predicate

$$\overline{x < 0} \to x = 0 \vee 0 < x,$$

we obtain

$$\vdash (\overline{0 < 0} \to 0 = 0 \vee 0 < 0) \,\&\, (x)\,[(\overline{x < 0} \to x = 0 \vee 0 < x) \to \\ \to (\overline{x' < 0} \to x' = 0 \vee 0 < x')] \to (\overline{y < 0} \to y = 0 \vee 0 < y). \tag{3}$$

The antecedent of this implication is the logical product of two true factors. The first of them,

$$\overline{0 < 0} \to 0 = 0 \vee 0 < 0$$

is deducible. In fact,

$$\vdash 0 = 0$$

holds, from which it follows that

$$\vdash 0 = 0 \vee 0 < 0;$$

and, finally, inasmuch as the deducibility of the implication follows from the deducibility of the consequent, we obtain

$$\vdash \overline{0 < 0} \to 0 = 0 \vee 0 < 0. \tag{4}$$

Furthermore, on the basis of the preceding theorem, we have

$$\vdash 0 < x',$$

from which it follows that

$$\vdash x' = 0 \vee 0 < x'.$$

And therefore

$$\vdash \overline{x' < 0} \to x' = 0 \vee 0 < x'.$$

From this we conclude that

$$\vdash (\overline{x < 0} \to x = 0 \vee 0 < x) \to (\overline{x' < 0} \to x' = 0 \vee 0 < x').$$

Finally, applying the product rule for binding by a quantifier, we obtain

$$\vdash (x)\,[(\overline{x < 0} \to x = 0 \vee 0 < x) \to (\overline{x' < 0} \to x' = 0 \vee 0 < x')]. \tag{5}$$

Formulae (4) and (5) represent the factors of the antecedent of formula (3); therefore, on the basis of the rule

$$\frac{A, B}{A \,\&\, B}$$

this antecedent is a deducible formula. By virtue of this, the consequent in formula (3) is also deducible and we have

$$\vdash \overline{y < 0} \rightarrow y = 0 \vee 0 < y.$$

We obtain the required formula by replacing y by x.

THEOREM 3. $\vdash \overline{x < 0}$.

Proof. Substituting the predicate $\overline{x < 0}$ for the predicate variable $A(x)$ in the complete induction axiom, we obtain

$$\vdash \overline{0 < 0} \,\&\, (x)\,[\overline{x < 0} \rightarrow \overline{x' < 0}] \rightarrow \overline{y < 0}. \tag{6}$$

The first factor is deducible by means of a substitution in axiom VII.1. Let us derive the second factor. By a substitution in axiom VII.2, we obtain

$$\vdash x < x' \rightarrow (x' < 0 \rightarrow x < 0).$$

The antecedent of this implication is axiom VII.3.

Applying the rule of inference, we have

$$\vdash x' < 0 \rightarrow x < 0.$$

Reversing the implication, we obtain

$$\vdash \overline{x < 0} \rightarrow \overline{x' < 0}.$$

Applying the product rule for binding by a quantifier, we have

$$\vdash (x)\,[\overline{x < 0} \rightarrow \overline{x' < 0}].$$

We have thus derived also the second factor of the consequent in formula (6). It follows from this that both the antecedent and consequent are deducible formulae, i.e.

$$\vdash \overline{y < 0}$$

and, consequently,

$$\vdash \overline{x < 0}.$$

COROLLARY. Combining Theorems 2 and 3 and applying the rule of inference, we obtain

$$\vdash x = 0 \vee 0 < x.$$

We have thus shown that it follows formally from the axioms of arithmetic that 0 is less than any other object of our system.

It is not difficult to show that no object of our system lies between the numerals 0 and $0'$, $0'$ and $0''$, and so on, i.e.

$$\vdash \overline{0 < x \,\&\, x < 0'},$$

$$\vdash \overline{0 < x \,\&\, x < 0''},$$

$$\dots\dots\dots\dots\dots\dots$$

$$\vdash \overline{0 < x \,\&\, x < 0^{(n)}},$$

It follows that $0'$ is the smallest of the objects which are not equal to 0, $0''$ is the smallest of the objects which are not equal to 0 and $0'$, and so on. In this system, the more general assertion

$$\overline{t < x \,\&\, x < t'}$$

is also deducible.

Furthermore, it can be proved (although we shall not do this here) that the system of axioms I-VIII possesses intrinsic completeness, i.e. all formulae which are true for an arbitrary interpretation of this system of axioms are deducible in it.

§7. Recursive terms

We shall now define a certain definite category of terms which we shall call *recursive terms*. We shall give this definition inductively. We list the initial recursive terms and give the rules for the formation of new recursive terms from those obtained earlier. Simultaneously with the rules for the formation of recursive terms we shall define rules for the formation of equalities between them and we shall take these equalities to be true or deducible *by definition*.

1. 0 is a recursive term.

2. Every object variable is a recursive term.

3. x' is a recursive term.

4. If $\mathfrak{a}$ is a recursive term containing the object variable x, then the term $\mathfrak{a}^*$ obtained from $\mathfrak{a}$ by means of a replacement of this object variable by an arbitrary recursive term is also a recursive term.

5. Suppose $\mathfrak{a}(x_1, \ldots, x_{n-1})$ is an arbitrary recursive term, which contains, of the object variables, only $x_1, \ldots, x_{n-1}$ but not necessarily all of these and perhaps none at all, and let $\mathfrak{y}(x_1, \ldots, x_{n+1})$ be an arbitrary recursive term containing, of the object variables, only $x_1, \ldots, x_{n+1}$, but again, perhaps, not all of them.

For every two such terms, we introduce a new recursive term which represents an object function containing all the variables $x_1, x_2, \ldots, x_n$ and only these. In this connection, every newly defined function must be denoted by a symbol which is distinct from the symbols of all the functions defined earlier, i.e. to two distinct pairs of terms one must set into correspondence distinct object functions.

We shall assume that to the pair of terms

$$\mathfrak{a}(x_1, \ldots, x_{n-1}) \text{ and } \mathfrak{y}(x_1, \ldots, x_{n+1})$$

is assigned the function $\xi(x_1, \ldots, x_n)$. We relate every such function with the corresponding terms with certain equations which we shall assume to be

deducible or true and which we shall call *recursive equations*. These equations have the following form:

$$
\begin{aligned}
&\xi(x_1, \ldots, x_{n-1}, 0) = a(x_1, \ldots, x_{n-1}),\\
&\xi(x_1, \ldots, x_{n-1}, x_n') = y(x_1, \ldots, x_n, \xi(x_1, \ldots, x_n)).
\end{aligned}
\tag{1}
$$

If it should turn out that the term y does not contain the variable x_{n+1} then we shall take the expression

$$y(x_1, \ldots, x_n, \xi(x_1, \ldots, x_n))$$

to represent the term y itself without any modification. For example, if $y(x_1, x_2)$ represents x_1', then the expression $y(x_1, \xi(x_1))$ also represents x_1'. Strictly speaking, the letters $x_1, \ldots, x_n$ which figure in the recursive equations do not denote arbitrary object variables—rather, they represent *definite* object variables, and, according to the definition, the recursive equations must be formulated with them. However, if instead of the recursive equations (1) we write the equations

$$
\begin{aligned}
&\xi(x, y, \ldots, u, 0) = a(x, y, \ldots, u),\\
&\xi(x, y, \ldots, u, v') = y(x, y, \ldots, u, \xi(x, y, \ldots, v)),
\end{aligned}
$$

which are obtained from the preceding equations by a substitution, then these equations will be true in the same calculus as were the former. We retain the name "recursive equations" for them and in the sequel, when necessary, we shall write them without any antecedents as basic recursive equations.

The rule for the formation of recursive terms and the rule for the formation of equations, described in section 5, above, apply also to the case $n = 1$. Only then the term a must not contain any object variable. The recursive equations in this case have the following form:

$$
\begin{aligned}
&\xi(0) = a,\\
&\xi(x_1') = y(x_1, \xi(x_1)).
\end{aligned}
$$

Thus, to the system of axioms VI, VII, VIII we adjoin an unlimited number of new true formulae which we called recursive equations. As a result we obtain a calculus which is stronger than the calculus with the system of axioms VI, VII, VIII, in the sense that the extended system describes the properties of the sequence of natural numbers incomparably more completely than the system of axioms VI, VII, VIII.

The calculus, which we have just obtained, containing axioms I-VIII, recursive terms and recursive equalities with the rules of inference of the extended predicate calculus will be called axiomatic arithmetic.

§8. Restricted arithmetic

If we delete from the axioms of axiomatic arithmetic the axiom of complete induction, we obtain a calculus which we shall call restricted arithmetic. We

shall call a recursive term which does not contain object variables a *recursive constant*. [This definition agrees with the definition of an object constant, given in §1.]

THEOREM. *For every recursive constant c there exists a numeral z such that the equality $c = z$ is deducible in restricted arithmetic.*

We shall prove our assertion in the following form.

Suppose c is an arbitrary recursive term and that c^* is a term obtained from c by an arbitrary replacement of all the numeral variables occurring in it. Then there exists a numeral z such that the equality $c^* = z$ is deducible in restricted arithmetic. The case when c contains no variables is contained in our result; then c^* coincides with c.

We proceed by induction, using the rules for the formation of recursive terms. Our assertion is obvious for the initial recursive terms because 0 is a numeral; a variable, after replacing it by a numeral, transforms into a numeral; the function x' also transforms into a numeral after replacing x by a numeral.

Suppose that for the recursive term $a(x)$, containing some variable (for example, x) and the recursive term k our assertion is valid. We shall show that it is then valid for the term $a(k)$ also. If we replace all variables in this term by numerals, then we obtain a term which can be represented in the form $a^*(k^*)$, where k^* is the result of replacing the variables in k by numerals and a^* is the result of replacing all the variables in $a(x)$, except x, by numerals. By hypothesis, for k^* there exists a numeral z^* such that the equation $k^* = z^*$ is deducible in restricted arithmetic.

But the implication

$$k^* = z^* \rightarrow a^*(k^*) = a^*(z^*)$$

is deducible from the axiom of equality, without an application of the axiom of complete induction (see §2); therefore, applying the rule of inference, we see that the equation

$$a^*(k^*) = a^*(z^*)$$

is also deducible in restricted arithmetic.

Since $a^*(z^*)$ represents the result of replacing all variables in $a(x)$ by numerals, then by assumption there exists a numeral z such that $a^*(z^*) = z$.

The equation

$$a^*(k^*) = z$$

is deducible from the equations

$$a^*(k^*) = a^*(z^*) \text{ and } a^*(z^*) = z$$

and the axiom of equality. Consequently, the equation $a^*(k^*) = z$ is deducible in restricted arithmetic. Since the term $a^*(k^*)$ is the result of an arbitrary replacement of the variables occurring in $a(k)$ by numerals, our assertion is proved for the term $a(k)$.

We suppose next that our assertion is valid for the term $\alpha(x_1, \ldots, x_{n-1})$ and the term $\gamma(x_1, \ldots, x_n, y)$ and we shall prove that it is then also valid for the term $\xi(x_1, \ldots, x_n)$, introduced on the basis of §5. We take $n-1$ arbitrary numerals $z_1, \ldots, z_{n-1}$ and consider the sequence of terms:

$$\xi(z_1, \ldots, z_{n-1}, 0),$$
$$\xi(z_1, \ldots, z_{n-1}, 0'), \ldots, \xi(z_1, \ldots, z_{n-1}, 0^{(k)}).$$

We shall prove that for every member of this sequence one can define the numeral $\overset{*}{z_k}$ such that the equality

$$\xi(z_1, \ldots, z_{n-1}, 0^{(k)}) = \overset{*}{z_k}$$

is deducible. We shall prove this by induction on the index k. The equation

$$\xi(z_1, \ldots, z_{n-1}, 0) = \alpha(z_1, \ldots, z_{n-1})$$

is true in restricted arithmetic. But, by hypothesis, for $\alpha(z_1, \ldots, z_{n-1})$ there exists a numeral $\overset{*}{z_0}$ such that the equation

$$\alpha(z_1, \ldots, z_{n-1}) = \overset{*}{z_0}$$

is deducible. Consequently, the equation

$$\xi(z_1, \ldots, z_{n-1}, 0) = \overset{*}{z_0}$$

is also deducible in restricted arithmetic.

We shall assume that for $\xi(z_1, \ldots, z_{n-1}, 0^{(k)})$ there exists a numeral $\overset{*}{z_k}$ such that the equality

$$\xi(z_1, \ldots, z_{n-1}, 0^{(k)}) = \overset{*}{z_k} \tag{1}$$

is deducible in restricted arithmetic. Moreover, on the basis of the assumption made concerning the function γ, there exists a numeral $\overset{*}{z}_{k+1}$ such that the equality

$$\gamma(z_1, \ldots, z_{n-1}, 0^{(k)}, \overset{*}{z_k}) = \overset{*}{z}_{k+1}$$

is deducible in restricted arithmetic.

Combining these two equations with the equation

$$\xi(x_1, \ldots, x_{n-1}, x_n') = \gamma(x_1, \ldots, x_n, \xi(x_1, \ldots, x_n)),$$

we obtain

$$\begin{aligned}\xi(z_1, \ldots, z_{n-1}, 0^{(k+1)}) &= \\ = \gamma(z_1, \ldots, z_{n-1}, 0^{(k)}, \xi(z_1, \ldots, z_{n-1}, 0^{(k)})) &= \\ = \gamma(z_1, \ldots, z_{n-1}, 0^{(k)}, \overset{*}{z_k}) &= \overset{*}{z}_{k+1}.\end{aligned}$$

Consequently, the equality

$$\xi(z_1, \ldots, z_{n-1}, 0^{(k+1)}) = \overset{*}{z}_{k+1}$$

is deducible in restricted arithmetic.

So, for every expression $\xi(z_1, \ldots, z_{n-1}, 0^{(k)})$, we can define a numeral $\overset{*}{z_k}$ such that equation (1) is deducible in restricted arithmetic.

Thus, our assertion is proved for all ways of forming recursive terms.

Since it is valid for the initial recursive terms, our inductive proof is complete, and we have proved that for every recursive constant we can define a numeral such that the equality of the given constant and the numeral is provable in restricted arithmetic.

If we assume that axiomatic arithmetic is consistent, *then for an arbitrary recursive constant c there exists a unique numeral z for which the equality $c = z$ is deducible in the restricted arithmetic.*

In fact, if two distinct numerals z_1 and z_2 could be found such that both $c = z_1$ and $c = z_2$ were deducible in restricted arithmetic, then the equation $z_1 = z_2$ would also be deducible in it. However, if z_1 and z_2 are distinct, then this equation leads to a contradiction in restricted arithmetic. In fact, suppose z_1 is $0^{(m)}$, that z_2 is $0^{(n)}$ and that the number of primes m is less than the number of primes n. Then, by substitutions in VII.3, we obtain

$$0^{(m)} < 0^{(m+1)}, 0^{(m+1)} < 0^{(m+2)}, \ldots, 0^{(n-1)} < 0^{(n)}, \ldots,$$

from which, applying axiom VII.2, we find

$$0^{(m)} < 0^{(n)}$$

is deducible from the axioms of group VII alone.

Consequently, $z_1 < z_2$ is deducible in restricted arithmetic. From the axioms of equality, we readily deduce

$$\overline{z_1 = z_2}$$

and, consequently, obtain a contradiction.

COROLLARY. *If a calculus containing restricted arithmetic is consistent and the equation $c_1 = c_2$, where c_1 and c_2 are recursive constants, is deducible in it, then this equation is deducible in restricted arithmetic.*

In fact, by what we have already proved there exists a numeral z_1 such that the equation $c_1 = z_1$ is deducible in restricted arithmetic. There also exists a numeral z_2 such that the equation $c_2 = z_2$ is deducible in restricted arithmetic. Since the calculus under consideration contains restricted arithmetic, the last two equations are also deducible in it. Therefore, the equation $z_1 = z_2$ is also deducible in it. But if the calculus is consistent, then the numerals z_1 and z_2 must coincide, otherwise the formula $\overline{z_1 = z_2}$ would be deducible in restricted arithmetic and consequently in the given calculus also. But if z_1 and z_2 represent the same numeral z and $c_1 = z$, $c_2 = z$ are deducible in restricted arithmetic, then $c_1 = c_2$ is also deducible in it. It also follows from the preceding remark that in the calculus under consideration for every recursive constant c there exists a unique numeral z such that the equality $c = z$ is deducible in this calculus.

In the sequel, for arbitrary recursive constants c_1 and c_2 we shall replace the expression "$c_1 = c_2$ is deducible in restricted arithmetic" by the shorter expression "c_1 is equal to c_2". The justification for omitting the expression

"restricted arithmetic" is that, as we saw, in an arbitrary consistent and stronger calculus the deducibility of the equality of recursive constants is equivalent to the deducibility of this equality in restricted arithmetic.

§9. Recursive functions

We now consider an arbitrary recursive term $g(x_1, \ldots, x_n)$. We assign in some way to each expression $g(z_1, \ldots, z_n)$ a definite numeral $h_{z_1, \ldots, z_n}$. We thus define on the domain of the numerals some function $h = h_{z_1, \ldots, z_n}$ which also takes numerals for values. This function, connected with the recursive term $g(x_1, \ldots, x_n)$, a symbol, has itself an informal mathematical meaning, namely, it is a function which, to each system of numerals $z_1, \ldots, z_n$, correlates a definite numeral $h_{z_1, \ldots, z_n}$. Such functions can be considered within the framework of mathematics in the same way as other mathematical concepts already known to us—for example, deducible formulae, the rules for obtaining true formulae and these or other correspondences which we have already considered for the symbols of the calculus.

We shall call the mathematical function defined by recursive terms in this way *recursive functions*. (More precisely, they ought to be called *primitive recursive functions* since recursive functions represent a more general concept.)

Addition and multiplication. We define the recursive term $\sigma(x, y)$ with the aid of the recursive equations

$$\text{(a)}\ \sigma(x, 0) = x,$$

$$\text{(b)}\ \sigma(x, y') = (\sigma(x, y))'.$$

To this term there corresponds the recursive function which coincides with the usual arithmetic sum in the sense that the value $\sigma(0^{(n)}, 0^{(m)})$ represents $0^{(n+m)}$. In view of this, we shall write the expression $\sigma(x, y)$ in the more customary form, namely, as $x + y$. Then the recursive equations for addition can be written as:

$$\text{(a)}\ x + 0 = x,$$

$$\text{(b)}\ x + y' = (x + y)'.$$

We now introduce the recursive term $\pi(x, y)$ by writing its recursive equations:

$$\text{(a)}\ \pi(x, 0) = 0,$$

$$\text{(b)}\ \pi(x, y') = \pi(x, y) + x.$$

The recursive function corresponding to the term $\pi(x, y)$ represents the usual arithmetic product. We shall employ the usual notation for it: $x.y$. Then the recursive equations which define it take the form

$$\text{(a)}\ x.0 = 0,$$

$$\text{(b)}\ x.y' = x.y + x.$$

Consider the recursive functions $\alpha(x)$ and $\beta(x)$:

$$\text{(a)}\ \alpha(0) = 0, \qquad \text{(a)}\ \beta(0) = 0',$$

$$\text{(b)}\ \alpha(x') = 0', \qquad \text{(b)}\ \beta(x') = 0.$$

As can be seen from the definition, the function $\alpha(z)$ equals 0 for $z = 0$ and it is equal to $0'$ for all other values of z.

The function $\beta(z)$ is on the contrary equal to $0'$ for $z = 0$ and it is equal to 0 for all other values of z.

Recursive functions $\delta(x)$ and $\delta(x, y)$ can be defined as follows:

$$\text{(a)}\ \delta(0) = 0,$$

$$\text{(b)}\ \delta(x') = x.$$

The function $\delta(z)$, defined by these formulae, obviously sets 0 into correspondence with 0 and every numeral, different from 0, is set into correspondence with the numeral preceding it. Thus, the function $\delta(z)$ formally corresponds to the subtraction of unity from all the numerals except 0.

$$\text{(a)}\ \delta(x, 0) = x,$$

$$\text{(b)}\ \delta(x, y') = \delta(\delta(x, y)).$$

We consider in more detail the recursive function defined by these equations:

$$\delta(x, 0') = \delta(\delta(x, 0)).$$

Consequently,

$$\delta(x, 0') = \delta(x).$$

Further,

$$\delta(x, 0'') = \delta(\delta(x, 0')),$$

i.e.

$$\delta(x, 0'') = \delta(\delta(x)).$$

It is easily seen that in general

$$\delta(x, 0^{(k)}) = \delta(\delta\underbrace{(\ldots\delta(x))\ldots)}_{k \text{ times}}.$$

So, $\delta(z_1, z_2)$ equals the numeral which is obtained from z_1 by application of the operation δ of one variable as many times as the numeral z_2 has primes above it. Thus, if z_1 is not less than z_2, then $\delta(z_1, z_2)$ is obtained by deleting from the numeral z_1 as many primes as the numeral z_2 contains. But if $z_1 < z_2$, then $\delta(z_1, z_2) = 0$. Stated more briefly,

$$\delta(0^{(k_1)}, 0^{(k_2)}) = 0^{(k_1-k_2)}$$

if $k_2 \leqslant k_1$ and

$$\delta(0^{(k_1)}, 0^{(k_2)}) = 0$$

otherwise.

§10. The axiomatic and formal deducibility of the properties of arithmetic functions

As we have already pointed out above, the concept of a recursive function, connected with a recursive term, is a formal concept. In particular, to the recursive terms $x + y$ and $x.y$ there correspond recursive functions which are essentially just the usual arithmetic sum and arithmetic product. Speaking more precisely, for every replacement of the variables x and y of the term $x + y$ by the numerals $0^{(i)}$ and $0^{(j)}$, the value of the corresponding recursive function is the numeral $0^{(i+j)}$, where here by $i + j$ is denoted the usual, informally understood, arithmetic sum of the number of primes in the first and second numerals, and the value of the recursive function corresponding to the term $x.y$ upon the replacement of x by the numeral $0^{(i)}$ and y by the numeral $0^{(j)}$ is the numeral $0^{(i.j)}$, where $i.j$ is the informally understood arithmetic product of the number of primes in the numerals $0^{(i)}$ and $0^{(j)}$. It is possible to deduce all arithmetical properties of the recursive functions corresponding to the terms $x + y$ and $x.y$ by purely formal arguments. To this end it is necessary simply to repeat the usual line of reasoning used for this purpose in common arithmetic. However, it must be noted that this type of proof does not consist of formal inferences *in* the restricted arithmetic itself but rather of formal arguments *about* the restricted arithmetic.

As an example, we consider the proposition: *The value of the recursive function corresponding to the term $x.y$ is* 0 *if the variable x or the variable y is replaced by the numeral* 0.

We shall prove this proposition by induction; not by means of an application of the axiom of complete induction, written in the form of a formula of the predicate calculus (for this has no part in restricted arithmetic) but rather by means of formal metalogical induction carried out on restricted arithmetic. Such an induction is an example of reasoning which occurs in the domain of finitism.

The fact that on replacement of the variable y by 0 the product $x.y$ takes on the value 0 follows directly from the definition of the recursive function $x.y$ by means of recursive equations. We shall show that on replacement of the variable x by the numeral 0 the recursive function also takes on the value 0. This assertion is true even if y also takes on the value 0. We shall assume that it is valid if y takes on the value z and prove that it is then valid on replacement of y by the numeral z'. In fact, by assumption, $0.z$ has the value 0. But $0.z'$ has, by definition, the same value as $0.z + 0$. But $0.z + 0$ has, by the definition of a sum, the same value as $0.z$, i.e. 0.

We have thus proved that the value of $0.z$ equals 0 for every numeral z. It is possible to prove all elementary theorems of arithmetic in an analogous manner. But—we repeat—all these theorems are not proved by the resources of restricted arithmetic itself but rather they are theorems about restricted arithmetic.

However, in axiomatic arithmetic with the axiom of complete induction it is possible to give proofs *in* the system of theorems corresponding to the informal theorems *about* arithmetic. Thus, for example, we can prove in axiomatic arithmetic the theorem

$$\vdash 0.y = 0 \mathbin{\&} x.0 = 0,$$

which corresponds to the informal theorem about restricted arithmetic we have just considered. We shall carry out this proof.

In order to prove the deducibility in arithmetic of the formula

$$\vdash 0.y = 0 \mathbin{\&} x.0 = 0,$$

it suffices to deduce

$$\vdash 0.y = 0$$

and

$$\vdash x.0 = 0.$$

The second of these formulae represents the first recursive equation connected with the term $x.y$ and it is therefore deducible in arithmetic.

We shall prove the deducibility of the first formula. Making a substitution in the axiom of complete induction of the formula $0.t = 0$ for $A(t)$, we obtain

$$\vdash 0.0 = 0 \mathbin{\&} (x)(0.x = 0 \rightarrow 0.x' = 0) \rightarrow 0.y = 0. \tag{1}$$

It follows from the first recursive equation for the term $x.y$ that

$$\vdash 0.0 = 0. \tag{2}$$

Then

$$\vdash 0.x = 0 \rightarrow 0.x = 0,$$

$$\vdash 0.x' = 0.x + 0,$$

$$\vdash 0.x + 0 = 0.x$$

hold. Replacing the term $0.x$ in the right member of the first of these formulae by the equal term $0.x + 0$ and taking the second equation into consideration, we obtain

$$\vdash 0.x = 0 \rightarrow 0.x' = 0.$$

Applying the rule for binding by a quantifier, we find that

$$\vdash (x)(0.x = 0 \rightarrow 0.x' = 0). \tag{3}$$

It follows from the deducibility of formulae (1), (2) and (3) that

$$\vdash 0.y = 0$$

and it is thus proved that

$$\vdash x.0 = 0 \mathbin{\&} 0.y = 0.$$

We note the difference between metalogical theorems about restricted arithmetic, in which properties of recursive functions are established, and the corresponding theorems of the arithmetic itself with the axiom of complete induction. The content of every metalogical theorem of the indicated type is that some formula

$$A(x_1, \ldots, x_n)$$

transforms into a formula which is deducible in restricted arithmetic for any replacement of the variables $x_1, \ldots, x_n$ by numerals. The content of the corresponding theorem of axiomatic arithmetic is that the formula

$$A(x_1, \ldots, x_n)$$

is deducible in arithmetic with the axiom of complete induction.

It may appear at first glance that these propositions are always equivalent. This, however, is not the case. The formal theorem of axiomatic arithmetic

$$\vdash A(x_1, \ldots, x_n)$$

is sometimes a stronger assertion than the assertion that every formula of the form

$$A(z_1, \ldots, z_n),$$

where $z_1, \ldots, z_n$ are arbitrary numerals, is deducible in restricted arithmetic. It occurs, for instance, when the formula $A(x_1, \ldots, x_n)$ has the form

$$r = 0, r_1 = r_2 \quad \text{or} \quad r_1 < r_2, \tag{4}$$

where r, r_1 and r_2 are arbitrary recursive terms. Now formal deducibility in arithmetic of any of formulae (4) implies the deducibility in the restricted arithmetic of the corresponding formula

$$r^0 = 0, r_1^0 = r_2^0 \quad \text{or} \quad r_1^0 < r_2^0,$$

obtained from the corresponding formula (4) by means of a replacement of all variables by numerals. In fact, if the formula $r_1 = r_2$ (respectively $r_1 < r_2$, $r = 0$) is formally deducible in arithmetic, then each of the formulae $r_1^0 = r_2^0$ (respectively $r_1^0 < r_2^0$, $r^0 = 0$), obtained from the preceding by a replacement of all variables by numerals is also formally deducible in arithmetic. But we know that for every recursive constant a numeral can be found which is equal to this constant in the sense that this equality is deducible in restricted arithmetic. From this, it follows that from the two formulae

$$r_1^0 = r_2^0 \text{ and } \overline{r_1^0 = r_2^0}$$

one is always deducible in restricted arithmetic. In exactly the same way, one of the formulae

$$r_1^0 < r_2^0 \text{ and } \overline{r_1^0 < r_2^0}$$

is deducible in restricted arithmetic.

But if the formula $r_1^0 = r_2^0$ is deducible in arithmetic and if arithmetic is consistent, then $r_1^0 = r_2^0$ must be deducible in restricted arithmetic also. The same can also be said about the formulae $r_1^0 < r_2^0$ and $r^0 = 0$. So, if a formula $A(x_1, \ldots, x_n)$ of the indicated type is deducible in arithmetic, then each of the formulae

$$A(z_1, \ldots, z_n)$$

is deducible in the restricted arithmetic.

The converse however does not hold. There exist formulae of the form

$r = 0$, where r is some recursive term, such that every formula of the form $r^0 = 0$, obtained by replacement in the formula $r = 0$ of all variables by numerals, is deducible in the restricted arithmetic whereas the formula $r = 0$ is not deducible in arithmetic despite the presence of the axiom of complete induction.

In the sequel we shall frequently make use of arithmetical properties of recursive functions in the non-formal metalogical sense. We shall, however, not prove these properties as the proofs would be a straightforward repetition of the corresponding arguments in arithmetic.

§11. Recursive predicates

Up to this point we used the expression "individual predicate" only for elementary expressions $P(x)$, $Q(x, y)$, $x = y$, ... We shall now generalize its meaning and we shall call every formula of arithmetic which contains object variables and does not contain predicate variables an *individual predicate. Individual predicates which are not elementary formulae will be called compound individual predicates.* We shall say that two predicates $P(x_1, \ldots)$ and $Q(x_1, \ldots)$ are equivalent if the formula obtained from the formula

$$P(x_1, \ldots) \sim Q(x_1, \ldots)$$

by the replacement of the free variable by arbitrary numerals is deducible in restricted arithmetic.

Predicates of the form

$$g(x_1, \ldots, x_n) = 0,$$

where g is a recursive term and all predicates equivalent to them are called *recursive predicates* corresponding to the term g.

THEOREM. *If all free variables of the recursive predicate $P_s(x_1, \ldots, x_n)$ are replaced by numerals $z_1, \ldots, z_n$, then either $P_s(z_1, \ldots, z_n)$ or $\bar{P}_s(z_1, \ldots, z_n)$ is deducible in restricted arithmetic.*

In fact, there exists a recursive term $k(x_1, \ldots, x_n)$ such that

$$P_s(z_1, \ldots, z_n) \sim k(z_1, \ldots, z_n) = 0$$

is deducible in restricted arithmetic. But we already know that it is possible to define a numeral z^* so that

$$k(z_1, \ldots, z_n) = z^*$$

is deducible in restricted arithmetic. Either $z^* = 0$ or $\overline{z^* = 0}$ is also deducible in restricted arithmetic. Consequently, either $P_s(z_1, \ldots, z_n)$ or $\bar{P}_s(z_1, \ldots, z_n)$ is deducible in restricted arithmetic, which is what we required to prove.

THEOREM. *If two predicates $P_s(x, y, \ldots)$ and $Q_s(x, y, \ldots)$ are recursive, then the predicates $P_s(x, y, \ldots) \vee Q_s(x, y, \ldots)$ and $\bar{P}_s(x, y, \ldots)$ are also recursive.*

Proof. If $P_s(x, y, \ldots)$ and $Q_s(x, y, \ldots)$ are recursive, then there exist recursive terms $\boldsymbol{k}$ and $\boldsymbol{q}$ such that P_s is equivalent to $\boldsymbol{k} = 0$ and Q_s is equivalent to $\boldsymbol{q} = 0$. We consider the term $\boldsymbol{k}.\boldsymbol{q}$. This is also a recursive term. We shall show that the predicate $\boldsymbol{k} = 0 \vee \boldsymbol{q} = 0$ is equivalent to the predicate $\boldsymbol{k}.\boldsymbol{q} = 0$. Indeed, in view of the fact that the recursive function corresponding to the term $\boldsymbol{k}.\boldsymbol{g}$ has all the properties of the usual arithmetic product, we can conclude that if $\boldsymbol{k}$ or $\boldsymbol{q}$ takes on the value 0, then $\boldsymbol{k}.\boldsymbol{q}$ also becomes 0 and conversely if $\boldsymbol{k}.\boldsymbol{q}$ takes on the value 0, then either $\boldsymbol{k}$ or $\boldsymbol{q}$ takes on the value 0. In this case the formula

$$\boldsymbol{k} = 0 \vee \boldsymbol{q} = 0 \sim \boldsymbol{k}.\boldsymbol{q} = 0$$

is deducible from the axioms of restricted arithmetic for every replacement by numerals and consequently the predicates

$$\boldsymbol{k} = 0 \vee \boldsymbol{q} = 0 \quad \text{and} \quad \boldsymbol{k}.\boldsymbol{q} = 0$$

are equivalent.

It is obvious that $\boldsymbol{k} = 0 \vee \boldsymbol{q} = 0$ is equivalent to $P_s \vee Q_s$. It follows from this that the last predicate is equivalent to $\boldsymbol{k}.\boldsymbol{q} = 0$ and consequently it is a recursive predicate.

To prove the recursiveness of the predicate $\bar{P}_s(x_1, \ldots)$, we consider the term $\beta(\boldsymbol{k})$, where β is the recursive term introduced above (see §9). It follows from the properties of this term that $\beta(\boldsymbol{k}) = 0$ if after the replacement of the free variables by numerals $\boldsymbol{k}$ is not equal to 0, and $\overline{\beta(\boldsymbol{k}) = 0}$ if $\boldsymbol{k} = 0$. But in this case $\beta(\boldsymbol{k}) = 0$ is equivalent to the predicate $\bar{P}_s(x_1, \ldots)$ and consequently this predicate is recursive; this is what we required to prove.

Utilizing the fact that the operations & and $\rightarrow$ of the propositional calculus are expressible in terms of the operations $\vee$ and —, it is easily shown that *if the predicates P_s and Q_s are recursive, then the predicates P_s & Q_s and* $P_s \rightarrow Q_s$ are also recursive.

Obviously, if the predicate $P_s(x_1, \ldots, x_n)$ is recursive, then the predicate $P_s(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ obtained from it by a replacement of the variables x_i by arbitrary recursive terms $\boldsymbol{q}_i$ is also recursive.

Recursiveness of the elementary predicates of arithmetic. We shall prove that *the predicates $x = y$ and $x < y$ are recursive.* In fact, we shall prove that the predicate $x = y$ is equivalent to the predicate

$$\delta(x, y) + \delta(y, x) = 0. \tag{1}$$

We substitute arbitrary numerals $0^{(n)}$ and $0^{(m)}$ for x and y. If $n = m$, then $\delta(0^{(n)}, 0^{(m)}) = 0$ and $\delta(0^{(m)}, 0^{(n)}) = 0$, and both the predicates $x = y$ and (1) are true. But if the numerals are not equal, then one of the terms in (1) equals 0 and the other is not 0. In this case, the sum is also different from 0. Consequently, the predicates $x = y$ and (1) are equivalent.

We can show in the same way that the predicate $x < y$ is equivalent to the predicate

$$\overline{\delta(y, x) = 0}.$$

But $\delta(y, x) = 0$ is a recursive predicate. The negation of a recursive predicate is also a recursive predicate. Consequently, $x < y$ is also a recursive predicate.

Starting from elementary predicates, one can, with the aid of the operations of the propositional calculus and substitutions, form new recursive predicates corresponding to variable recursive terms.

§12. Other methods of forming recursive predicates: bounded quantifiers

Let $P_s(x)$ be some recursive predicate. Then, by definition, there exists a recursive term $f(x)$ such that, for every numeral z, the formula

$$P_s(z) \sim f(z) = 0$$

is deducible in restricted arithmetic.

In connection with the predicate $P_s(x)$, we introduce the recursive term $\nu(x)$ with the recursive equations

$$\nu(0) = f(0)$$
$$\nu(x') = \nu(x).f(x').$$

This recursive term $\nu(h)$ will also be expressed conditionally in the form

$$\bigcap_{x \leqslant h} f(x).$$

Such a representation is convenient because (as can easily be proved by induction) $\nu(z)$ equals the arithmetic product:

$$f(0).f(0') \ldots f(z).$$

To the term $\bigcap\limits_{x \leqslant h} f(x)$ there corresponds the recursive predicate

$$\bigcap_{x \leqslant h} f(x) = 0,$$

which we shall denote by the symbol

$$\underset{\leqslant h}{(\exists x)} P_s(x).$$

For every numeral z, the predicate $\underset{\leqslant z}{(\exists x)} P_s(x)$ is equivalent to the formula

$$f(0).f(0') \ldots f(z) = 0$$

and, consequently, to the formula

$$P_s(0) \vee P_s(0') \vee \ldots \vee P_s(z).$$

The symbol

$$\underset{\leqslant h}{(\exists x)}$$

will be called the *bounded existential quantifier*.

We shall also introduce the symbol for the *bounded universal quantifier:*

$$\underset{\leqslant h}{(x)}.$$

The expression

$$\underset{\leqslant h}{(x)} P_s(x)$$

will be defined to be equivalent to the expression

$$\overline{\underset{\leqslant h}{(\exists x)} \overline{P_s(x)}}.$$

The predicate, represented by this expression, is also a recursive predicate.

Clearly the formula

$$\underset{\leqslant z}{(x)} P_s(x),$$

is equivalent to the formula

$$P_s(0) \;\&\; P_s(0') \;\&\; \ldots \;\&\; P_s(z).$$

§13. Methods for forming new recursive terms

We shall now introduce some auxiliary methods enabling us to form recursive terms, starting from recursive predicates. Suppose $P_s(x)$ is a recursive predicate containing the variable x (and, perhaps, other variables) and let $f(x)$ be the recursive term corresponding to it. We define a new recursive term

$$\min_{0<x\leqslant h} P_s(x)$$

which can also be written as $g(x_1, \ldots, x_k, h)$, where $x_1, \ldots, x_k$ are free variables distinct from x which occur in $P_s(x)$. We write down the defining equations for the term $\min\limits_{0<x\leqslant h} P_s(x)$ in the usual form:

(a) $\min\limits_{0<x\leqslant 0} P_s(x) = 0.$

(b) $\min\limits_{0<x\leqslant h'} P_s(x) = \min\limits_{0<x\leqslant h} P_s(x) + h'\beta[\min\limits_{0<x\leqslant h} P_s(x) + f(h')].$

Clearly, there exists no number x satisfying the inequality $0 < x \leqslant 0$. The expression $\min\limits_{0<x\leqslant 0} P_s(x)$, however, has meaning and is the designation of the term we introduced $\min\limits_{0<x\leqslant z} P_s(x)$ for $z = 0$. We shall show that for every numeral z, different from 0, we have that

$$\min_{0<x\leqslant z} P_s(x) = \boldsymbol{h},$$

where $\boldsymbol{h}$ is the smallest numeral which is distinct from 0 and not exceeding z for which the predicate $P_s(x)$ represents a true formula, and 0 if such a numeral does not exist. Our assertion is obvious if $z = 0$ inasmuch as there do not exist numerals which are different from 0 and not exceeding 0. We shall assume that our assertion is valid for $z = 0^{(k)}$ and show that in this case it is also valid for the numeral $0^{(k+1)}$. On the basis of condition (b), above, we have that

$$\min_{0<x\leqslant 0^{(k+1)}} P_s(x) = \min_{0<x\leqslant 0^{(k)}} P_s(x) + 0^{(k+1)}\,\beta[\min_{0<x\leqslant 0^{(k)}} P_s(x) + f(0^{(k+1)})].$$

We shall assume that there exists a numeral $0^{(j)}$, different from 0 and not exceeding $0^{(k)} (j \leqslant k)$, for which $P_s(0^{(j)})$ is true. Then among such numerals a smallest one can be found; suppose this is $0^{(j)}$ itself. Then

$$\min_{0<x\leqslant 0^{(k+1)}} P_s(x) = 0^{(j)} + 0^{(k+1)} \cdot \beta[0^{(j)} + f(0^{(k+1)})].$$

But $0^{(j)} + f(0^{(k+1)})$ is greater than 0, and, therefore,

$$\beta[0^{(j)} + f(0^{(k+1)})] = 0$$

and, consequently,

$$\min_{0<x\leqslant 0^{(k+1)}} P_s(x) = 0^{(j)};$$

and our assertion is proved for this case.

We now assume that among the numerals $0', 0'', \ldots, 0^{(k)}$ there is none for which $P_s(x)$ is true. Then, by assumption,

$$\min_{0<x\leqslant 0^{(k)}} P_s(x) = 0$$

and, consequently,

$$\min_{0<x\leqslant 0^{(k+1)}} P_s(x) = 0^{(k+1)}\ \beta[f(0^{(k+1)})].$$

If $P_s(0^{(k+1)})$ is a true formula, then $f(0^{(k+1)}) = 0$ and $\beta(0) = 0'$. Then

$$\min_{0<x\leqslant 0^{(k+1)}} P_s(x) = 0^{(k+1)}.0' = 0^{(k+1)},$$

which agrees with our assertion.

If $P_s(0^{(k+1)})$ is false, then

$$\overline{f(0^{(k+1)}) = 0}$$

is true. In this case,

$$\beta[f(0^{(k+1)})] = 0$$

and

$$\min_{0<x\leqslant 0^{(k+1)}} P_s(x) = 0.$$

And so our assertion is valid in this case also.

We define analogously the function

$$\max_{0<x\leqslant h} P_s(x),$$

where $P_s(x)$ is a recursive predicate:

(a) $\max\limits_{0<x\leqslant 0} P_s(x) = 0,$

(b) $\max\limits_{0<x\leqslant h'} P_s(x) = \max\limits_{0<x\leqslant h} P_s(x).\alpha(f(h')) + h'\beta(f(h')),$

and where $f(x)$ is a recursive term corresponding to the predicate $P_s(x)$. We shall show that

$$\max_{0<x\leqslant z} P_s(x)$$

is the largest of those numerals

$$0', 0'', \ldots, 0^{(k)} = z$$

for which $P_s(x)$ is true, and that

$$\max_{0<x\leqslant z} P_s(x) = 0$$

if such a numeral does not exist.

Our assertion is obviously valid for $z = 0$. We shall assume that it is valid for the numeral $0^{(k)}$. We shall show that it is then true for $0^{(k+1)}$ also. First suppose that among the numerals $0', \ldots, 0^{(k)}$ there are numerals for which $P_s(x)$ is true and that $0^{(j)}$ is the largest of these numerals. In this case,

$$\max_{0<x\leqslant 0^{(k)}} P_s(x) = 0^{(j)}$$

and

$$\max_{0<x\leqslant 0^{(k+1)}} P_s(x) = 0^{(j)}.\alpha(f(0^{(k+1)}) + 0^{(k+1)}\cdot\beta(f(0^{(k+1)})).$$

If $P_s(0^{(k+1)})$ is true, then

$$f(0^{(k+1)}) = 0$$

and

$$\alpha(f(0^{(k+1)})) = 0,$$

$$\beta(f(0^{(k+1)})) = 0'.$$

Then

$$\max_{0<x\leqslant 0^{(k+1)}} P_s(x) = 0^{(k+1)}.$$

But if $P_s(0^{(k+1)})$ is false, then, conversely:

$$\alpha(f(0^{(k+1)})) = 0',$$

$$\beta(f(0^{(k+1)})) = 0$$

and

$$\max_{0<x\leqslant 0^{(k)}} P_s(x) = 0^{(j)}.$$

In both cases our assertion is valid.

We shall now assume that among the numerals $0', \ldots, 0^{(k)}$ there is no numeral for which $P_s(x)$ is true. In this case, by definition,

$$\max_{0<x\leqslant 0^{(k)}} P_s(x) = 0$$

and

$$\max_{0<x\leqslant 0^{(k+1)}} P_s(x) = 0^{(k+1)}.\beta(f(0^{(k+1)})).$$

If $P_s(0^{(k+1)})$ is true, then

$$f(0^{(k+1)}) = 0$$

and

$$\beta(f(0^{(k+1)})) = 0'.$$

Then

$$\max_{0<x\leqslant 0^{(k+1)}} P_s(x) = 0^{(k+1)}.$$

But if $P_s(0^{(k+1)})$ is false, then

$$\beta[f(0^{(k+1)})] = 0$$

and

$$\max_{0<x\leqslant 0^{(k+1)}} P_s(x) = 0.$$

So, our assertion is proved for all cases. Consequently, it is valid for all numerals.

§14. Some number-theoretical predicates and terms

1. *The predicate of divisibility $D_s(x, y)$.* We shall denote by $D_s(x, y)$ the predicate expressed by the formula

$$\underset{\leqslant y}{(\exists t)}[t.x = y].$$

It is easily seen that if $D_s(z_1, z_2)$ is true, then there exists a numeral h such that

$$h.z_1 = z_2,$$

i.e. z_2 *is divisible by* z_1. In fact, we shall assume that $z_1 = 0^{(k_1)}$ and that $z_2 = 0^{(k_2)}$. The formula

$$\underset{\leqslant z_2}{(\exists t)}[t.z_1 = z_2]$$

is equivalent to the formula

$$0.z_1 = z_2 \vee 0'.z_1 = z_2 \vee \ldots \vee z_2.z_1 = z_2.$$

By assumption, this formula is true. Since the sum is true, at least one of the equations occurring in it must be true, i.e. a numeral h can be found such that

$$h{\cdot}z_1 = z_2.$$

We shall now assume that $D_s(z_1, z_2)$ is false. Then the formula

$$\underset{\leqslant z_2}{(t)}\, \overline{[t{\cdot}z_1 = z_2]},$$

or, what amounts to the same thing, the formula

$$\overline{0.z_1 = z_2}\ \&\ \overline{0'.z_1 = z_2}\ \&\ \ldots\ \&\ \overline{z_2.z_1 = z_2}$$

holds. But among the numerals $0, 0', \ldots, z_2$ no numeral h can be found such that $h.z_1 = z_2$ is a true formula. But in this case there is in general no numeral satisfying this equation. In fact, in the case when $z_1 = 0$ and $\overline{z_2 = 0}$, it is clear that no numeral h exists for which $h{\cdot}z_1 = z_2$. If z_2 is not 0, then $z_2 < h{\cdot}z_1$ if h is larger than z_2.

2. *The predicate "to be a prime number" $P_r(x)$.* We shall denote by $P_r(x)$ the predicate expressed by the formula

$$\underset{\leqslant x}{(t)}\, [0' < x \,\&\, (0' < t < x \rightarrow \overline{D_s(t, x)})]. \qquad (*)$$

[$0' < t < x$ is the abbreviated way of writing the predicate $0' < t \,\&\, t < x$.] It is clear from the formula (*), above, that $P_r(x)$ is a recursive predicate. It is easy to establish (analogously to example 1) that for those, and only those, numerals z for which $P_r(z)$ is a true formula, there does not exist a numeral, different from $0'$ and from z, which divides z; we suppose that z is greater than $0'$.

The function $k!$ This function is defined by the following equations;

$$\text{(a) } 0! = 0';$$

$$\text{(b) } h'! = h'.h!$$

It is easy to establish that

$$z! = 0'\cdot 0'' \ldots z$$

holds for every numeral z.

In the sequel, we shall need the fact that "for every numeral z, different from 0, there exists a numeral r among the numerals $z, z', \ldots, z! + 0'$ such that $P_r(r)$ is true in restricted arithmetic". In order to prove this fact, it is sufficient to rely on Euclid's familiar proof. This proof does not use the concept of actual infinity since the argument refers to the finite domain of numerals up to $z! + 0'$. Moreover, all concepts employed are represented by recursive predicates which are constructively calculable for numerals.

3. *The recursive predicate $S_r(x, y, z)$ having the meaning "x is a prime number lying between y' and z".* This predicate is defined by the formula

$$P_r(x) \mathbin{\&} y < x \leqslant z.$$

It follows from this formula that $S_r(x, y, z)$ is a recursive predicate. We now define the recursive term $\phi(x, y)$ by the following formula:

$$\phi(x, y) = \min_{0<t\leqslant y} S(t, x, y)$$

The recursive function, defined by this term, is the smallest prime number between the numbers x' and y if it exists, and 0 otherwise.

4. *The recursive function $\pi(x)$ whose values range, successively, over all prime numbers.* We define the recursive term $\pi(x)$ by the following equalities:

(a) $\pi(0) = 0''$.

(b) $\pi(x') = \phi(\pi(x), \pi(x)! + 0')$.

We shall show that the values of the recursive function $\pi(z)$ range, successively, over all prime numbers. By virtue of (a), above, the value of $\pi(z)$ for $z = 0$ is equal to the smallest prime number. We shall assume that

$$\pi(0), \pi(0'), \ldots, \pi(z)$$

represents the sequence of successive prime numbers (without omissions). But

$$\pi(z') = \phi(\pi(z), \pi(z)! + 0'),$$

i.e. the value of $\pi(z')$ is the smallest prime number comprised between $\pi(z)$ and $\pi(z)! + 0'$, or 0 if there is no such number. But, by Euclid's theorem, such a number exists—and, consequently, the value of $\pi(z')$ is the prime number which follows immediately after $\pi(z)$. So, $\pi(z)$ is the recursive function sought.

5. *The predicate "x is index of the power of a prime divisor of the number y in the resolution of this number into prime factors" $R_s(x, y)$.*

This predicate is expressed by the formula

$$\underset{\leqslant y}{(\exists t)}[P_r(t) \mathbin{\&} D(t^x, y) \mathbin{\&} \overline{D(t^{x'}, y)}].$$

It is clear from the formula that the predicate $R_s(x, y)$ is recursive.

6. *The predicate $W_s(x)$ "x is a number of the form*

$$p_1^{h_1} \cdot p_2^{h_2} \cdots p_k^{h_k}\text{"}, \tag{1}$$

where p_i is the ith prime (in order); h_i is the power of this prime factor in the product (in particular, h_i can be equal to 0): k is a fixed number.

This predicate is defined by the formula

$$\underset{\leqslant x}{(\exists u_1)} \ldots \underset{\leqslant x}{(\exists u_k)}[p_1^{u_1} \cdot p_2^{u_2} \ldots p_k^{u_k} = x].$$

This is obviously a recursive predicate.

7. *The recursive function $\sigma(x)$ which ranges over all numbers of the form*

$$p_1^{u_1} \cdot p_2^{u_2} \cdots p_k^{u_k}$$

for fixed k. The construction of this function is analogous to the construction of a function which ranges over all prime numbers. We shall first construct the recursive predicate $M_s(x, y, z)$, corresponding to the phrase: "x is a number of the form

$$p_1^{u_1} \cdots p_k^{u_k},$$

lying between the numbers y' and z". $M_s(x, y, z)$ is expressed by the formula

$$W_s(x) \mathbin{\&} y < x \leqslant z$$

and is obviously a recursive predicate. We introduce the recursive term $\psi(x, y)$ in the following way:

$$\psi(x, y) = \min_{0 < t \leqslant y} M_s(t, x, y).$$

We now write the equations which define the required recursive function:

(a) $\sigma(0) = p_1^0 \cdot p_2^0 \cdots p_k^0$,

(b) $\sigma(x') = \psi(\sigma(x), 0''.\sigma(x))$.

The function thus defined ranges successively over all numbers of the form (1). In fact, $\sigma(0)$ is the minimal number of the form (1). Further, it follows from (b) that if $\sigma(z)$ is a number of the form (1), then $\sigma(z')$ is the smallest number of the form (1), lying between $(\sigma(z))'$ and $0''.\sigma(z)$ if such a number exists. Since $0'' = p_1$, it follows that $0''.\sigma(z)$ is a number of the form (1), and therefore there exists a number of the form (1) between $(\sigma(z))'$ and $0''.\sigma(z)$; consequently, $\sigma(z')$ is the number of the form (1) which follows immediately after $\sigma(z)$.

8. We construct k recursive functions $\rho_1(x), \ldots, \rho_k(x)$ such that when z

ranges successively over all numerals, then the set of values of the k functions $\rho_1(z), \ldots, \rho_k(z)$ ranges over all possible groups of k numerals. We set

$$\rho_1(z) = \max_{0<t\leqslant z} D(p_1^t, \sigma(z));$$

$$\rho_2(z) = \max_{0<t\leqslant z} D(p_2^t, \sigma(z));$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\rho_k(z) = \max_{0<t\leqslant z} D(p_k^t, \sigma(z)).$$

The first of these functions, $\rho_1(z)$, is the greatest power of the prime p_1, in the factorization of the number $\sigma(z)$; the second function, $\rho_2(z)$, is the greatest power of the prime p_2, in the factorization of $\sigma(z)$; and so on, until, finally, $\rho_k(z)$ is the greatest power of the prime number p_k in the factorization of $\sigma(z)$. Since $\sigma(z)$ ranges over all numbers of the form (1) when z ranges successively over all numerals, the set of values of the functions $\rho_1(z), \ldots, \rho_k(z)$ defined in this way ranges over all possible groups of k numerals when z ranges successively over all numerals.

§15. Calculable functions

The adjunction of recursive equations to axioms VI, VII, VIII introduces arithmetic operations on terms. If we limit ourselves only to recursive functions defining addition, multiplication and raising to a power, then we obtain a system of axioms which represents an inessential modification of the Peano system of axioms. The adjunction of all recursive equations introduces into arithmetic—besides the usual arithmetic operations—an extensive class of other operations. As we have seen, with the aid of these operations all possible number-theoretical concepts connected with the divisibility of numbers, prime factors, powers of prime factors, and many others can be defined.

Recursive functions connected with recursive terms represent a rather extensive class of functions defined on the sequence of natural numbers and taking integer values (numeral values in our terminology).

Furthermore, the definition of these functions is such that they are effectively calculable, i.e. for every set of values of the arguments of the function $r(x_1, \ldots, x_n)$, we find, in an effective way, the value of the function itself. The concept of an *effectively calculable* function is of very great significance. The general concept of an algorithm reduces to it. If an algorithm performs definite operations in some regular fashion, then one can easily imagine that the elements of these operations and all possible combinations of them can be denoted by numerals such that every act of the operation and its result are indexed by a definite numeral. Then that which is accomplished by the algorithm will appear as a successive construction of numerals $z_1, z_2, \ldots, z_n, \ldots$. Thus, to the algorithm there corresponds some

calculable function defined on numerals and taking numerals also as values. The assertion that every algorithm can be described in the form of a calculable function of one variable should not appear surprising since a calculable function of an arbitrary number of variables can be reduced to a calculable function of one variable. If we have, for example, a calculable function of n variables $r(x_1, \ldots, x_n)$, we enumerate all groups of n numerals by means of the recursive functions

$$z_1 = \rho_1(z), \ldots, z_n = \rho_n(z),$$

so that to each group $z_1, \ldots, z_n$ we set into correspondence an index z. The function

$$r(\rho_1(z), \ldots, \rho_n(z))$$

sets into correspondence to the index of the group $(z_1, \ldots, z_n)$, in a calculable way, the value of the function r which it takes on for the group $z_1, \ldots, z_n$.

The concept of "algorithm" has existed and has been used in mathematics for a long time. However, our understanding of it was of a purely intuitive character, which, naturally, hampered the use of this concept. A number of authors (Church, Turing, Post, and others) gave definitions of the concept of algorithm, all of which were different in form, but which turned out to be mutually equivalent. This definition is formulated most simply in terms of calculable functions. Recursive functions, which we considered in preceding sections, are calculable. We might have supposed that the intuitive concept of a calculable function coincides with the concept of recursive function. It turns out, however, that this is not the case. We can define a function of two variables $v(x, h)$, possessing the following properties:

1. For every pair of numerals z_1, z_2 the value $v(z_1, z_2)$ of the function represents a numeral, in a calculable way, which corresponds completely to our idea of an algorithm.

2. For every recursive function $r(z_1)$ there exists a numeral z_2^0 such that $r(z_1)$ equals $v(z_1, z_2^0)$ for every value of z_1.

In this case, the function $v(z, z) + 0'$ is, obviously, a calculable function. We shall show that it does not coincide with a single recursive function. In fact, we shall assume that $v(z, z) + 0'$ represents a recursive function. We denote it by $c(z)$. In this case, a numeral $\dot{z}_2$ can be found such that $c(z_1)$ equals $v(z_1, \dot{z}_2)$ for every value of z_1. For z_1 equal to $\dot{z}_2$, the value of the function $c(\dot{z}_2)$ equals $v(\dot{z}_2, \dot{z}_2)$. But $c(\dot{z}_2)$ represents $v(\dot{z}_2, \dot{z}_2) + 0'$; consequently, $v(\dot{z}_2, \dot{z}_2)$ equals $v(\dot{z}_2, \dot{z}_2) + 0'$, which is impossible. Thus, assuming that $v(z, z) + 0'$ is a recursive function, we have arrived at a contradiction.

So, it is impossible to hold that the concept of a recursive function coincides with the concept of a calculable function. However, the definition of a calculable function is closely connected with the concept of a recursive function. The notion of a *general recursive function* is introduced. In distinction to general recursive functions, the recursive functions introduced above are frequently called *primitive recursive functions*. We have not used

this term so far because we had not introduced the concept of a general recursive function. In the sequel it will also be unnecessary to use the term as we shall restrict ourselves to general recursive functions. Therefore, in the following chapter we shall continue to call primitive recursive functions—recursive fnnctions.

The definition of a general recursive function is as follows. Let $r(z_1, z_2)$ be an arbitrary primitive recursive function of two variables and suppose that $f(z_2)$ is an arbitrary primitive recursive function of one variable. We shall assume that for every value of z_1 there exists a unique numeral z_2 such that

$$\vdash r(z_1, z_2) = 0$$

holds.

Then this relation defines an implicit function $g(z_1)$ such that

$$\vdash z_2 = g(z_1) \sim r(z_1, z_2) = 0.$$

Furthermore, we can also define the function

$$f(g(z_1)).$$

We shall call a function general recursive if it coincides with the function $f(g(z_1))$ *formed from some pair of primitive recursive functions* $r(z_1, z_2)$ *and* $f(z)$ *in the manner indicated.*

As we pointed out above, by the term "calculable function" we understand that for this function there is given an algorithm which allows one, for every value of the argument, to define the value of the function. But the problem consists in sharpening the concept of a calculable function, since the concept of algorithm, on which it is based, is used here for the time being only in an intuitive sense, without any definition whatsoever. The definition of a calculable function is formulated in the following way.

A calculable function is a general recursive function. In this connection, the law for the calculation of the values of a general recursive function consists in that for every numeral z we calculate the value of the primitive recursive functions $r(z, 0)$, $r(z, 0')$, and so on, until we arrive at a numeral z^* such that $r(z, z^*)$ takes on the value 0. By virtue of our assumption about the function r, this must eventually happen and, consequently, after a finite number of steps, we obtain the required numeral z^*. After this, we calculate the value of a second recursive function $f(z^*)$ which, by definition, will be the value of the given general recursive function. It may appear at first glance that the definition of a calculable function just introduced is of too special a character and does not correspond to our intuitive idea of an algorithm and of a calculable function. In other words, one might think that it is possible to find an algorithm for calculating the values of some function which is known not to be general recursive. However, it turns out that all methods, known to us, for constructing calculable functions lead to general recursive functions. A more detailed analysis of this problem, which we shall not embark on, convincingly shows that the concept of a general recursive function

encompasses our intuitive idea of a calculable function and includes the most general concept of an algorithm. Since we are concerned with the comparison of an exact definition of a general recursive function with an intuitive idea of a calculable function, there cannot be any talk of rigorous proof.

The definition of the concept of an algorithm has been used to prove the non-existence of an algorithm for the solution of one or another class of problems. Thus, for example, Church showed that there does not exist an algorithm solving the decision problem for the predicate calculus. Problems which consist in finding an algorithm which solves one or other infinite series of problems of one type are called *algorithmic problems.* The non-solvability of an algorithmic problem means that the algorithm sought does not exist. In recent times, a number of results has been obtained concerning the non-solvability of algorithmic problems in various branches of mathematics. In particular, the Soviet mathematicians A. A. Markov, P. S. Novikov, and their students established the non-solvability of a number of algorithmic problems which touch upon groups, subgroups, matrices, polyhedra, and so on. These results are applications of mathematical logic to problems lying outside it.

§16. Some theorems of axiomatic arithmetic

In arithmetic, represented by the system of axioms I-VIII with recursive equations, all known theorems of elementary arithmetic are formally deducible. Apparently, in it are also deducible all theorems known at the present time in the theory of numbers. The fact that the theory of numbers extensively uses the means and ideas of analysis only proves that in these means are comprised rich heuristic elements which allow one to find approaches and paths for the solution of difficult problems in number theory. But, furthermore, it is entirely possible that the formal proof of any theorem in number theory can be carried out by means of axiomatic arithmetic alone. We shall here restrict ourselves to a formal deduction of the most fundamental theorems of elementary arithmetic—namely, we shall deduce the properties of addition and multiplication.

THEOREM 1. $\vdash 0 + x = x$.

Proof. We perform a substitution in the axiom of complete induction, replacing $A(t)$ by $0 + t = t$; we obtain

$$\vdash 0 + 0 = 0 \,\&\, (x)\,[0 + x = x \to 0 + x' = x'] \to 0 + y = y. \tag{1}$$

It follows from the recursive equations which define the sum function that

$$\vdash 0 + 0 = 0. \tag{2}$$

Further, on the basis of the general property of uniqueness of terms (see §2), we have

$$\vdash 0 + x = x \to (0 + x)' = x'.$$

From the recursive equations defining a sum it follows that

$$\vdash (0 + x)' = 0 + x'),$$

from which, replacing in the preceding formula the term $(0 + x)'$ by the term $0 + x'$ which is equal to it, we obtain

$$\vdash 0 + x = x \to 0 + x' = x'.$$

Applying the derived rule for binding by a quantifier, we have

$$\vdash (x)(0 + x = x \to 0 + x' = x'). \tag{3}$$

It follows from the truth of formulae (2) and (3) that the antecedent in formula (1) is true. Therefore, applying the rule of inference, we obtain

$$\vdash 0 + y = y,$$

and the theorem is proved.

THEOREM 2. $\vdash z + x' = z' + x$.

Proof. We substitute $z + t' = z' + t$ in place of $A(t)$ in the axiom of complete induction. We obtain that

$$\vdash z + 0' = z' + 0 \,\&\, (x)[z + x' = z' + x \to z + x'' = \\ = z' + x'] \to z + y' = z' + y. \tag{4}$$

On the basis of the recursive equations, we have

$$\vdash z + 0' = (z + 0)',$$

$$\vdash z + 0 = z,$$

$$\vdash z' + 0 = z'.$$

Replacing equal terms in these equations, we obtain

$$z + 0' = (z + 0)' = z' = z' + 0,$$

from which it follows that

$$\vdash z + 0' = z' + 0. \tag{5}$$

We take the formula

$$z + x' = z' + x \to (z + x')' = (z' + x)', \tag{6}$$

which is true in arithmetic. By virtue of the recursive equations, we have

$$\vdash (z + x')' = z + x'';$$

$$\vdash (z' + x)' = z' + x'.$$

Replacing equal terms by equals in formula (6), we obtain

$$\vdash z + x' = z' + x \to z + x'' = z' + x'. \tag{7}$$

From the truth of formulae (5) and (7) follows the truth of the antecedent and this means also the truth of the consequent of formula (4):

$$\vdash z + y' = z' + y.$$

The theorem just proved can be formulated as follows: *If in a sum a prime from one term is carried over to another, then the term determined by the sum goes over into an equal sum.*

THEOREM 3. $\vdash z + x = x + z.$

Proof. Substituting $z + t = t + z$ in place of $A(t)$ in the axiom of complete induction, we obtain

$$\vdash z + 0 = 0 + z \mathbin{\&} (x)[z + x = x + z \rightarrow z + x' = \\ = x' + z] \rightarrow z + y = y + z. \tag{8}$$

From the recursive equations defining a sum, it follows that

$$\vdash z + 0 = z.$$

On the basis of Theorem 1, we have

$$\vdash 0 + z = z.$$

From the last two equations, we obtain

$$\vdash z + 0 = 0 + z. \tag{9}$$

From the property of the uniqueness of terms, we have

$$\vdash z' + x = x + z \rightarrow (z + x)' = (x + z)', \tag{10}$$

and from the recursive equations

$$\vdash (z + x)' = z + x',$$

$$\vdash (x + z)' = x + z',$$

we obtain, on the basis of Theorem 2, that

$$\vdash x + z' = x' + z$$

and, consequently, that

$$\vdash (x + z)' = x' + z.$$

Replacing equal terms by equals in formula (10), we obtain:

$$\vdash z + x = x + z \rightarrow z + x' = x' + z. \tag{11}$$

From the truth of formulae (9) and (11) follows the truth of the antecedent and hence of the consequent of formula (8) also:

$$\vdash z + y = y + z.$$

We have thus proved the *commutativity of addition.*

THEOREM 4. $(z + u) + x = z + (u + x).$

Proof. Substituting $(z + u) + t = z + (u + t)$ in place of $A(t)$ in the axiom of complete induction, we obtain

$$\vdash (z + u) + 0 = z + (u + 0) \mathbin{\&} (x)((z + u) + x = z + (u + x) \rightarrow \\ \rightarrow (z + u) + x' = z + (u + x')) \rightarrow (z + u) + y = z + (u + y). \tag{12}$$

From the recursive equations, we have

$$\vdash (z + u) + 0 = z + u;$$

$$\vdash u + 0 = u.$$

It follows from the second equation that

$$\vdash z + (u + 0) = z + u.$$

It follows easily from these equations that

$$\vdash (z + u) + 0 = z + (u + 0). \tag{13}$$

From the property of the uniqueness of terms, it follows that

$$\vdash (z + u) + x = z + (u + x) \rightarrow ((z + u) + x)' = (z + (u + x))'. \tag{14}$$

From the recursive equations, we deduce the following formulae:

$$\vdash ((z + u) + x)' = (z + u) + x';$$

$$\vdash (z + (u + x))' = z + (u + x)';$$

$$\vdash (u + x)' = u + x'.$$

It follows from the last two equations that

$$\vdash (z + (u + x))' = z + (u + x').$$

Replacing equal terms by equals in formula (14), we obtain

$$\vdash (z + u) + x = z + (u + x) \rightarrow (z + u) + x' = z + (u + x').$$

Binding the last formula by a quantifier:

$$\vdash (x)[(z + u) + x = z + (u + x) \rightarrow (z + u) + x' = z + (u + x')], \tag{15}$$

we conclude from the truth of formulae (13) and (15) that the antecedent and hence the consequent of formula (12) also are true, i.e.

$$\vdash (z + u) + y = z + (u + y).$$

We have thus proved the *associativity of addition.*

For multiplication one can also deduce the laws of associativity, commutativity, and distributivity with respect to addition. The formulae which express these laws have the usual form in our notation:

$$\vdash (x.y).z = x.(y.z);$$

$$\vdash xy = yx;$$

$$\vdash (x + y)z = xz + yz.$$

We shall restrict ourselves to the proof of the last formula, assuming the first two have already been derived.

THEOREM 5. $\vdash (x + y).z = x.z + y.z.$

Proof. Replacing $A(x)$ by the formula $(u + v)x = ux + vx$ in the axiom of complete induction, we obtain

$$\vdash (u + v).0 = u.0 + v.0 \,\&\, (x)[(u + v)x = ux + vx \rightarrow$$
$$\rightarrow (u + v)x' = ux' + vx'] \rightarrow (u + v).y = uy + vy. \tag{16}$$

From the recursive equations which define multiplication, it follows that

$$\vdash (u + v).0 = 0;$$

$$\vdash u.0 = 0;$$

$$\vdash v.0 = 0.$$

Moreover,

$$\vdash 0 + 0 = 0$$

holds. It follows from the last four equations that

$$\vdash (u + v).0 = u.0 + v.0. \tag{17}$$

We now note that if

$$\vdash s = t$$

holds, then

$$\vdash s + w = t + w$$

also holds. In fact, replacing in the true formula

$$\vdash s + w = s + w$$

in the right member the term s by the term t equal to it, we obtain

$$\vdash s + w = t + w.$$

Performing a substitution in the true formula

$$\vdash s = t \rightarrow s + w = t + w,$$

we obtain

$$\vdash (u + v).x = u.x + v.x \rightarrow (u + v).x + (u + v) = \\ = (ux + vx) + (u + v). \tag{18}$$

On the basis of the associative and commutative laws for addition, we have

$$\vdash (ux + vx) + u + v = (ux + u) + (vx + v).$$

Replacing the last term in formula (18) by an equal, we obtain

$$\vdash (u + v).x = ux + vx \rightarrow (u + v)x + (u + v) = \\ = (ux + u) + (vx + v). \tag{19}$$

On the basis of the recursive equations defining the sum function, we deduce that

$$\vdash (u + v)x' = (u + v)x + (u + v);$$

$$\vdash ux' = ux + u;$$

$$\vdash vx' = vx + v.$$

Replacing certain terms in formula (19) by equals, we have

$$\vdash (u + v)x = ux + vx \rightarrow (u + v)x' = ux' + vx'.$$

Binding the last formula by a quantifier, we obtain

$$\vdash (x)((u + v)x = ux + vx \rightarrow (u + v)x' = ux' + vx'). \tag{20}$$

We conclude from the truth of formulae (17) and (20) that the antecedent and hence the consequent of formula (16) also are true:

$$(u + v)y = u.y + v.y.$$

The law of the distributivity of multiplication with respect to addition is thus proved.

We have already pointed out that the system of axioms VI, VII, VIII, without recursive functions, fails to express all properties of the natural numbers since such basic properties of natural numbers, as, for example:

$$x' = y' \rightarrow x = y$$

are not deducible in it.

The property of total order is also not deducible:

$$\overline{x = y} \rightarrow x < y \vee y < x,$$

together with many others. The introduction of recursive functions rectifies this situation, and in arithmetic with recursive functions all the fundamental properties of the sequence of natural numbers turn out to be deducible.

Here we shall restrict ourselves to the deduction of the first of the properties mentioned above, and for the second we shall only briefly note the idea of the proof.

THEOREM 6. $\vdash x' = y' \rightarrow x = y$.

Proof. We write the recursive equations for the function $\delta(x)$ defined in §9:

$$\vdash \delta(0) = 0;$$

$$\vdash \delta(x') = x.$$

On the basis of the property of the uniqueness of terms, we have

$$\vdash x = y \rightarrow \delta(x) = \delta(y).$$

Performing substitutions in this formula, we obtain

$$\vdash x' = y' \rightarrow \delta(x') = \delta(y'). \tag{21}$$

On the basis of the recursive equations for the function $\delta(x)$, we obtain

$$\vdash \delta(x') = x';$$

$$\vdash \delta(y') = y.$$

Replacing terms by equal terms in formula (21), we will have

$$\vdash x' = y' \rightarrow x = y.$$

The proof of the formula

$$\overline{x = y} \rightarrow x < y \vee y < x \tag{22}$$

can be carried out in the following manner. First, with the aid of the axiom of complete induction, we may prove that

$$\vdash x < y \sim (\exists t)(y = x + t').$$

If we denote the formula $(\exists t)(y = x + t)$ by $A(x, y)$, then the problem of proving formula (22) reduces to the problem of proving the formula

$$\overline{x = y} \rightarrow A(x, y) \vee A(y, x).$$

This formula is proved without any difficulty with the aid of the axiom of complete induction.

CHAPTER VI

ELEMENTS OF PROOF THEORY

§1. Formulation of the problems of consistency and independence of axioms

In Chapter II we described a method which was and is used to prove the consistency and independence of axioms. This method of interpreting a system of axioms on some set of objects, constructed by means of set theory, is taken as basic. But the fact that a set-theoretic basis is unsatisfactory is at the same time a fundamental reason for turning to the axiomatic description of mathematical systems. There therefore arose another formulation of the problems of independence and consistency. We have already discussed this several times. Here we shall only recall the fundamental idea in the formulation of the problems of the consistency and independence of axioms. To prove the intrinsic consistency of a calculus means to prove that it does not contain a formula A such that it and its negation $\bar{A}$ are both deducible in the calculus. The independence of an axiom means that it cannot be deduced from the other axioms by means of the deduction rules in the calculus under consideration. It is not necessary to resort to an interpretation in order to solve the problem of the consistency of a calculus or the independence of any one of its axioms. It is required, by metalogical means, to prove the impossibility of the deduction in it of formulae. A new formulation of the problems of the consistency and independence of axioms called for new methods for the solution of these problems. These methods constitute so-called *proof theory*. We shall become acquainted with them in applications to problems of axiomatic arithmetic. We pose the problem of finding a method of solving the following two problems:

(1) the problem of the consistency of restricted arithmetic, and

(2) the problem of the independence of the axiom of complete induction in arithmetic.

As far as the problem of the consistency of arithmetic with the axiom finite induction is concerned, there arise difficulties of so fundamental a nature that the metalogical methods we have been using turn out to be insufficient for the solution of this problem. At the present time, problems connected with this take up an important place in mathematical logic. We shall stop to consider them in somewhat more detail.

In Chapter IV, page 201, we mentioned the fact that it is impossible to prove the consistency of any calculus by methods which are formalizable in the calculus itself. The precise meaning of this statement consists in the following.

If one formalizes methods by means of which the consistency of a calculus is proved, then the system obtained as a result of this formalization will contain formulae which are not deducible in the calculus whose consistency is being proved. This result is of very general application and it applies to every axiomatic arithmetic.

The finitistic metalogic we have assumed can be so formalized that all propositions in it are formulae of axiomatic arithmetic and all arguments are formal deductions in this arithmetic. It turns out, therefore, to be impossible to prove the consistency of axiomatic arithmetic by means of finitistic metalogic. A change in the very formulation of the problem of the consistency of axiomatic arithmetic is therefore required. If we are dealing with foundations for the idea of actual infinity, then the problem of the consistency of arithmetic can be formulated in a sufficiently satisfactory way. The analysis of the foundations of mathematics made by Brouwer showed that the only principle in arithmetic which depends on actual infinity is the law of the excluded middle. If we assume arithmetic without the law of the excluded middle to be consistent, then its consistency with the law of the excluded middle can be proved. [See A. N. Kolmogorov, *On the tertium non datur principle*, Matem. Sbornik, Volume 32 (1925), pp. 646-667, and K. Gödel, *Zur intuitionistischen Arithmetik und Zahlentheorie*, Ergebnisse eines math. Koll., Heft 4 (for 1931-1932, published in 1933), pp. 34-38.]

We shall not touch upon the problem of the consistency of axiomatic arithmetic in this book. We shall limit ourselves to the solution of certain consistency problems for which Hilbert's finitistic methods are sufficient.

In particular, we shall prove the consistency of restricted arithmetic. Although the solution of this problem is a quite incomplete and restricted result, it is none the less of interest since we prove with the aid of a formal system of reasoning, not depending on the concept of actual infinity, the consistency of a system which has as its subject matter an infinite set of objects and allows for them such means of reasoning as the law of the excluded middle.

Thus, the proof of the consistency of restricted arithmetic is the basis of the possibility of using actual infinity to some extent.

§2. Prime factors and prime summands

We shall assume that the formula A is a logical product, i.e. that it has the form $A_1 \& \ldots \& A_n$. It may happen that certain (and perhaps even all) factors are also products. Suppose, for example, that A_1 has the form

A_{11} & ... & A_{1m}. In virtue of the associative law, formula A is equivalent to the formula

$$A_{11} \mathbin{\&} \ldots \mathbin{\&} A_{1m} \mathbin{\&} A_2 \mathbin{\&} \ldots \mathbin{\&} A_n.$$

We can continue to make similar transformations until we arrive at a formula which is equivalent to formula A and is a product, none of whose factors are products. We shall call the factors of such a logical product *prime factors.* We see that every logical product can be transformed, in virtue of the associative law, into a product which is equivalent to it in which all factors are prime. We shall call the indicated transformation a *resolution into prime factors.* It is perfectly clear that the resolutions of products of prime factors obtained in the result is absolutely independent of the order in which we carry out the resolution. Thus, the structure of the prime factors is uniquely determined by the original formula A. (It is easily seen that the prime factors themselves are parts of the formula A.) In virtue of this, we can speak of the prime factors of a product, understanding by this those prime factors which can be obtained as the result of the resolution of the formula A.

In an analogous manner, we call a term in a logical sum which is itself not a sum a *prime summand.* Reasoning as in the case of a product, one can show that every logical sum can be resolved into prime summands. This resolution is also unique so that it makes sense to speak of the prime summands of an arbitrary formula A, which is a sum.

It is easily seen that every non-prime factor A_i of the product A itself resolves into prime factors which occur among the prime factors of the formula A. And, conversely, every prime factor of the product A_1 & ... & A_n either coincides with one of the factors A_i or is a prime factor of one of these factors. Analogously, every non-prime term of a sum resolves into prime terms which occur among the prime summands of the given sum. In all our further discussion, an arbitrary product A and a product A' which is the resolution of A into prime factors are absolutely equivalent so that A can always be replaced by A'. Analogously, an arbitrary sum can be replaced by its resolution into prime summands.

§3. Primitively true formulae

In the sequel we introduce the important concept of a *regular formula.* But we must first give certain preliminary definitions.

A formula which does not contain quantifiers is called a primitive formula.

We consider all elementary individual propositions and predicates occurring in a primitive formula. Every proposition of this sort has the form $r_1 = r_2$ or $r_1 < r_2$, where r_1 and r_2 are certain constants.

Every elementary individual predicate has the same form except that now r_1 and r_2 are terms of which at least one contains object variables. If in an elementary individual predicate we replace the object variables by numerals, we obtain an elementary proposition of the indicated type. If r_1 and r_2 are

recursive constants then, as we showed in Chapter V (§8), numerals can be assigned to them in a unique and completely finitist manner. Let these numerals be z_1 and z_2, respectively.

An elementary individual proposition is said to be primitively true if the number of primes on the numerals z_1 *and* z_2 *are the same in the case when the proposition has the form* $r_1 = r_2$ *and the number of primes on* z_1 *is less than the number of primes on* z_2 *in the case when the elementary proposition has the form* $r_1 < r_2$. Otherwise, we shall call these elementary propositions *primitively false.*

We can state the following proposition.

If an elementary proposition of the indicated type is primitively true, then it is deducible in restricted arithmetic. The validity of this assertion follows from the description of recursive functions (see Chapter V).

In a primitive formula, besides individual elementary formulae, there can also occur propositional variables $A, B, \ldots$ and predicate variables $P(x)$, $Q(x, y)$, $F(0, z)$, and so on. We replace all non-recursive predicates by recursive predicates, then all object variables by numerals, and, finally, all primitively true propositions obtained by the symbol T and primitively false propositions by the symbol F. The remaining elementary formulae are replaced by the symbols T and F in an arbitrary manner but so that the same expressions are replaced in the same way. As a result of this we obtain a formula which can be considered as a formula in propositional algebra, which, by the rules of the propositional algebra, can now be uniquely assigned the value T or F. Which of these two values the formula assumes can be determined in a finite number of steps.

A primitive formula is called a primitively true formula if:

1^0. *It is deducible in restricted arithmetic.*

2^0. *After replacing the variable predicates by arbitrary recursive predicates and the object variables by arbitrary numerals, and replacing elementary formulae by the symbols T and F in the manner indicated above, it becomes a formula in propositional algebra, always taking the value T.*

In the case when the primitive formula satisfies condition 2^0 only, it will be called *primitively true in the weak sense.*

We note that if one assumes the consistency of restricted arithmetic, then condition 1^0 implies condition 2^0. However, we shall not make this assumption inasmuch as we are considering the problem of proving the consistency of restricted arithmetic. In the sequel, we shall make essential use of property 2^0 of a primitively true formula. We have, therefore, defined it explicitly.

EXAMPLES.

1. $A(x) \vee \bar{A}(x) \mathbin{\&} 0 < x'$.

If we replace x by the numeral $0^{(k)}$, we obtain the formula

$$A(0^{(k)}) \vee \bar{A}(0^{(k)}) \mathbin{\&} 0 < 0^{(k)\prime}.$$

But $0^{(k)\prime}$ is the numeral $0^{(k+1)}$. The individual proposition $0 < 0^{(k+1)}$ is primitively true and is therefore replaced by the symbol T. We then obtain the formula

$$A(0^{(k)}) \vee \bar{A}(0^{(k)}) \mathbin{\&} T.$$

Considering $A(0^{(k)})$ as a propositional variable in propositional algebra, we obtain an identically true formula. Consequently, formula 1 is primitively true.

2. The axioms of arithmetic in groups VI and VII are primitively true formulae.

VI. Equality axioms:

$$1.\ x = x,$$

$$2.\ x = y \rightarrow (A(x) \rightarrow A(y)).$$

VI.1, clearly, is primitively true. Replacing x and y in VI.2 by the numerals z_1 and z_2, we obtain the formula

$$z_1 = z_2 \rightarrow (A(z_1) \rightarrow A(z_2)).$$

If the numerals z_1 and z_2 are distinct, then we must assign the value F to the expression $z_1 = z_2$ and therefore every formula assumes the value T for arbitrary values of $A(z_1)$ and A(z_2). If z_1 coincides with z_2, we obtain the formula

$$z = z \rightarrow (A(z) \rightarrow A(z)).$$

The second member in this implication is an identically true formula in the sense of propositional algebra; consequently, the formula also assumes the value T. Thus, formula VI.2 is also primitively true.

VII. Order axioms:

$$1.\ \overline{x < x},$$

$$2.\ x < y \rightarrow (y < z \rightarrow x < z),$$

$$3.\ x < x'.$$

It is easily verified that in every replacement of x, y, z in these axioms by numerals we obtain formulae which assume the value T. This fact is verified in a finite number of steps. We can therefore assert that all the axioms in group VII are primitively true formulae.

The axiom of complete induction is not a primitive formula, and, consequently, cannot be primitively true.

3. We consider an example of a primitive formula which is not primitively true:

$$(A(x) \rightarrow A(y)) \rightarrow x = y.$$

We replace x by the numeral 0 and y by the numeral $0'$. This yields

$$(A(0) \rightarrow A(0')) \rightarrow 0 = 0'.$$

We replace $A(0)$ by F and $A(0')$ by T; we must replace $0 = 0'$ by the symbol F. We then obtain the formula

$$(F \rightarrow T) \rightarrow F.$$

This formula assumes the value F. Consequently, the initial primitive formula is not primitively true.

We note that if we carry out identical transformations of propositional algebra on a primitively true formula, the result is also a primitively true formula.

§4. The operations 1, 2, 3

We introduced the concept of a reduced formula in the logic of predicates (see Chapter III). Let us recall this definition.

A reduced formula is a formula which does not contain the symbol $\rightarrow$ and which is such that the negation symbols in it apply only to elementary parts. This definition can also be generalized to formulae of arithmetic. The basic fact that for every formula there exists a reduced formula which is equivalent to it can also be generalized. The proof of this in arithmetic remains the same as in predicate calculus. A reduced formula which is equivalent to a given formula will be called the reduced form of the latter. All further definitions will be with respect to reduced formulae.

We now introduce the concept of a *regular formula* which will be very important in the sequel.

An arbitrary reduced formula can be represented in the following form:

$$(x_1) \ldots (x_n)(A_1 \vee \ldots \vee A_n) \,\&\, (B_1 \vee \ldots \vee B_m) \,\&\, \ldots$$
$$\ldots \,\&\, (C_1 \vee \ldots \vee C_v). \quad (\alpha)$$

It may be assumed that arbitrary variables can occur here in place of the variables $x_1, \ldots, x_n$, that there may be no quantifiers $(x_1), \ldots, (x_n)$ at all, that the product appearing after the quantifier symbols $(x_1), \ldots, (x_n)$ may reduce to a single term, and, finally, that every sum

$$A_1 \vee \ldots \vee A_n,\ B_1 \vee \ldots \vee B_m,$$

and so on, may also reduce to a single term. In these cases we shall agree to say that we have a product consisting of one factor or a sum consisting of one summand.

The quantifiers $(x_1), \ldots, (x_n)$ will be called *exterior quantifiers*. The prime factors of the product appearing after the quantifier symbols

$$(x_1), \ldots, (x_n)$$

will be called *exterior factors*. If the product appearing after the quantifier symbols contains non-prime factors, then we can always resolve it into prime factors; such a transformation has no effect at all on our further discussion. In virtue of this, we can assume that all factors in formula (α) are prime. In

exactly the same way, each of these prime factors which is a sum can be resolved into simple summands and we can assume that all the terms $A_i, B_j, \ldots, C_k$ are prime. The prime summands of the factors in formula (α) are called *exterior summands* of formula (α). We note further that if the product appearing after the quantifier symbols $(x_1), \ldots, (x_n)$ consists of one factor, which in turn consists of one summand, then we can assume that this factor does not have the form $(x)A(x)$ inasmuch as in the contrary case we can put it with the exterior quantifiers.

We introduce operations which we will perform on formulae which are representable in the form (α).

1. First operation. If a universal quantifier of a prime summand of an exterior factor (for example, A_1) has the form $(z)A_1'(z)$ then (z) is taken outside and becomes an exterior quantifier. In this connection, the variable connected with this quantifier is renamed if it coincides with other variables in the formula. We call this operation *displacement of a universal quantifier.*

If the quantifier (z) of the factor A_1 is displaced, then formula (α) takes on the form

$$(x_1) \ldots (x_n)(z)(A_1'(z) \vee \ldots \vee A_n) \mathbin{\&} (B_1 \vee \ldots) \mathbin{\&} \ldots .$$

It may happen that the term $A_1'(z)$ is no longer prime. This would not matter but for the sake of convenience one can at once resolve the sum

$$A_1'(z) \vee \ldots \vee A_n$$

into prime summands.

2. Second operation. This operation is connected with a term of the form $(\exists z)A_1'(z)$. Suppose, for example, that A_1 has such a form. The operation consists in adjoining a term of the form $A_1'(t)$ to the sum

$$A_1 \vee \ldots \vee A_n,$$

where t is an arbitrary recursive term containing arbitrary object variables, besides variables, which are interrelated in the formula

$$(A_1 \vee \ldots \vee A_n) \mathbin{\&} \ldots \mathbin{\&} (C_1 \vee \ldots \vee C_p)$$

(thus, the term t can contain the variables $x_1, x_2, \ldots, x_n$).

If this operation is performed on the term A_1, then, as a result, we obtain the formula

$$(x_1) \ldots (x_n)((\exists z)A_1(z) \vee A_1(t) \vee A_2 \vee \ldots \vee A_n) \mathbin{\&} \ldots \\ \ldots \mathbin{\&} (C_1 \vee \ldots \vee C_p).$$

The new term may not be prime and one can again resolve it into prime summands. We call this operation *separation from the existential quantifier.*

3. Third operation. This is a transformation of the distributivity of logical addition with respect to multiplication. This operation is applied when one of the terms in formula (α) is a product and, besides, *is not a primitive formula.* Suppose, for example, that A_1 has the form of the

product $A_{11} \& \ldots \& A_{1r}$ (where we can assume that the factors A_{1i} are prime). The third operation consists in resolving an exterior factor which contains this term into factors

$$(A_{11} \vee A_2 \vee \ldots \vee A_n) \& (A_{12} \vee \ldots \vee A_n) \& \ldots \& (A_{1r} \vee \ldots \vee A_n)$$

and formula (α) goes over into the formula

$$(x_1) \ldots (x_n)(A_{11} \vee A_{12} \vee \ldots \vee A_n) \& \ldots$$
$$\ldots \& (A_{1r} \vee A_2 \vee \ldots \vee A_n) \& (B_1 \vee \ldots \vee B_m) \& \ldots.$$

Again, as above, non-prime summands A_{1i} can appear in the new factors; it is convenient to resolve them so that we have to deal only with sums all of whose summands are prime in all further discussion.

We note that the third operation *is not applied to primitive summands.* This means that in our case the summand A_1 must contain at least one quantifier. This operation is called a *distributive operation.*

§5. Regular formulae

We now give the definition of a regular formula.

1. *A formula is called elementary regular if it is primitively true or if it has the form of a disjunction one of whose terms is primitively true.*

Formula (α) *is called regular if each of its exterior factors is elementary regular or if it can be reduced to this form by means of operations* 1, 2, 3.

We introduce also a somewhat different concept which we shall call *weak regularity.*

A formula is called elementary regular in the weak sense if it has the form $A \vee B$, *where* A *is a formula which is primitively true in the weak sense and* B *is an arbitrary formula.*

A formula is called weakly regular if it can be reduced by means of operations 1, 2, 3 *to a formula all exterior factors of which are elementary regular in the weak sense.*

All the lemmas proved in the subsequent sections remain valid if we replace in them the phrase "regular formula" by the phrase "weakly regular formula". In this connection, the proofs are unchanged or in some cases are simplified.

We now give some examples of regular formulae.

1. $(\exists x)A(x) \vee \bar{A}(y)$.

Here, the product in formula (α) reduces to a single factor and exterior quantifiers are absent. Applying the second operation to this formula, we obtain the formula

$$A(y) \vee (\exists x)A(x) \vee \bar{A}(y),$$

which is elementary regular.

2. $(\exists x)((y)A(y) \vee \bar{A}(x))$.

There are no exterior quantifiers in this formula either. The product reduces to a single factor and this factor reduces to a single summand. We apply the second operation and obtain

$$(y)A(y) \vee \bar{A}(0) \vee (\exists x)((y)A(y) \vee \bar{A}(x)).$$

Further, we apply the first operation—i.e. we displace the first quantifier (y) and rename the variable connected with it:

$$(z)(A(z) \vee \bar{A}(0) \vee (\exists x)((y)A(y) \vee \bar{A}(x))).$$

We apply the second operation to this formula once more and obtain:

$$(z)(A(z) \vee \bar{A}(0) \vee (y)A(y) \vee \bar{A}(z) \vee (\exists x)((y)A(y) \vee \bar{A}(x))).$$

The single factor in this formula is elementary regular since it contains the primitively true summand $A(z) \vee \bar{A}(z)$. Consequently, formula 2 is regular.

3. $(x)(\exists y)(y = \phi(x))$.

Applying the second operation—i.e. replacing the variable y by the term $\phi(x)$ in the isolated member—we obtain:

$$(x)(\phi(x) = \phi(x) \vee (\exists y)(y = \phi(x))).$$

The formula after the quantifier symbol (x) is elementary regular. Consequently, formula 3 is regular.

4. $(x)(y)((\bar{A}(x) \vee (\exists z)A(z)) \,\&\, (A(y) \vee \bar{A}(y) \,\&\, x < x'))$.

We perform the second operation on the quantifier $(\exists z)$:

$$(x)(y)((\bar{A}(x) \vee A(x) \vee (\exists z)A(z)) \,\&\, (A(y) \vee \bar{A}(y) \,\&\, x < x')).$$

We now perform the third operation on the second factor:

$$(x)(y)((\bar{A}(x) \vee A(x) \vee (\exists z)A(z)) \,\&\, (A(y) \vee \bar{A}(y)) \,\&\, (A(y) \vee x < x')).$$

In this formula, every factor is elementary regular and, consequently, formula 4 is regular.

In order to prove the regularity of any formula, we apply operations 1, 2, 3 and obtain a succession of formulae:

$$A_0, A_1, \ldots, A_n = A.$$

Every formula A_{i-1} is obtained from formula A_i by application of one of the operations 1, 2, 3. If, as a result, formula A_0 is obtained, all of whose exterior factors are elementary regular, then formula A is regular.

The series $A_0, A_1, \ldots, A_n$, in which every formula A_{i-1} is obtained from formula A_i by application of one of the operations 1, 2, 3, and every exterior factor of formula A_0 is elementary regular, is called a regularity series of formula A_n.

Obviously, every formula of a regularity series is a regular formula. In the sequel, we shall prove certain properties of regular formulae by applying

induction with respect to regularity series. The scheme of reasoning in this connection is the following: the property is proved for the formula A_0 all exterior factors of which are elementary regular. Then, for an arbitrary regularity series, it is proved that if the assertion is true for A_{i-1} then it is also true for A_i. From this we conclude that the property holds for any regular formula. We set ourselves the problem of proving that *every formula which is deducible in restricted arithmetic is regular.* But we must first prove a series of auxiliary propositions.

Operations 1, 2, 3 possess the property that the changes which they effect in the exterior factors of an arbitrary formula A do not depend on what the exterior quantifiers in the formula A are (or whether there are any in the first place).

We shall assume that some formula A is regular. Let

$$K_0, K_1, \ldots, K_n$$

be its regularity series (where K_n coincides with A). We consider the formula A' which differs from A only in its exterior quantifiers, i.e. it can be obtained from A by deletion of exterior quantifiers or by the prescription of new ones.

In this case, performing these same operations on A' as were performed on A, we obtain the regularity series $K'_0, \ldots, K'_n$, where K'_n coincides with A'.

The exterior factors of the formulae K'_i and K_i are the same. Consequently, the exterior factors of the formulae K'_0 and K_0 are also the same. Therefore, since the exterior factors of K_0 are elementary regular by assumption, the exterior factors of K'_0 are also elementary regular. Then, by the definition of regularity, the formula K'_n, i.e. A', is also regular. It follows from this that:

1. *If the formula A' is regular and has the form $(x)A(x)$, then the formula $A(x)$ is also regular,*

and, conversely:

2. *If the formula $A(x)$ is regular, then the formula $(x)A(x)$ is also regular.*

In fact, (x) is an exterior quantifier in the formula $(x)A(x)$ and therefore its elimination does not disturb regularity; in exactly the same way, the adjunction of an exterior quantifier to the regular formula $A(x)$ does not disturb regularity.

Let us consider the following arbitrary regularity series:

$$K_0, K_1, \ldots, K_n.$$

If we eliminate all the exterior quantifiers in all the formulae in this series, we obtain the series

$$K'_0, \ldots, K'_n,$$

whose formulae K'_i are products of exterior factors of the formulae in the preceding series (i.e. of the K_i). However, we determine the regularity of the

formulae K_i (as well as of the K_i') from the second series as well as from the first as this property is completely determined by the exterior factors alone.

It is easily seen that the formulae $K_n', K_{n-1}', \ldots, K_0'$ are obtained, one after the other, by the same operations as were the formulae of the first series—with the exception of the operation of removing a quantifier. This operation is replaced by the operation of eliminating a quantifier—of course, with the corresponding change in the variable which it bounds.

We shall call this operation the elimination of the quantifier of operation 1′. Thus, the regularity series

$$K_0, K_1, \ldots, K_n$$

can be transformed into the series

$$K_0', K_1', \ldots, K_n',$$

by eliminating all exterior quantifiers of the formula K_n and applying—further—the same operations as for the first series, with the exception of operation 1, which is replaced by operation 1′. Since the formulae of the first and second series differ only by the presence of exterior universal quantifiers and, on the other hand, K_0' is the product of exterior factors of the formula K_0, it follows from the preceding discussion that the regularity of the first series implies the regularity of the second series, and conversely. Thus, in the definition of regularity, we can replace the operations 1, 2, 3 by the operations 1′, 2′, 3′. We retain the name regularity series for the series

$$K_0', \ldots, K_n'.$$

We shall prove some properties of regular formulae.

1. *Every exterior factor of a regular formula is itself a regular formula.*

Let A be a regular formula. Applying operations 1, 2, 3, we obtain for it the regularity series

$$K_0, K_1, \ldots, K_n,$$

where K_n is the formula A.

We consider the corresponding regularity series

$$K_0', \ldots, K_n'$$

which is obtained from K_n' by means of the operations 1′, 2′, 3′.

We consider an arbitrary exterior factor of the formula A, for example $A^{(1)}$. $A^{(1)}$ is an exterior factor of the formula K_n' also. We denote it by $A_n^{(1)}$. We know that the operations 1, 2, 3 are always performed on one of the exterior factors. The same is obviously true of the operation 1′. It may happen that an operation performed on the formula K_n' is *not* applied to the factor $A_n^{(1)}$. Then $A_n^{(1)}$ occurs in K_{n-1}' without modification. And it may happen that the operation *is* applied to this factor. Then it transforms into another factor or a product of factors, which will appear in the formula K_{n-1}'.

We denote by $A_{n-1}^{(1)}$ the factor or product of factors that the factor $A_n^{(1)}$ goes over into.

In the transition from the formula K_{n-1}' to K_{n-2}' a factor which does not occur in the composition of $A_{n-1}^{(1)}$ can undergo modification. Then $A_{n-1}^{(1)}$ goes over into K_{n-2}' without modification. In the contrary case, $A_{n-1}^{(1)}$ is changed and goes over into the formula $A_{n-2}^{(1)}$, all of whose factors will be exterior factors of the formula K_{n-2}'. Continuing this process further, we obtain the series

$$A_n^{(1)}, A_{n-1}^{(2)} \ldots, A_0^{(1)}.$$

In this series, certain terms can be repeated. Eliminating the superfluous terms, we obtain the series

$$A_n^{(1)}, A_{n-s_1}^{(1)}, \ldots, A_{n-s_k}^{(1)}$$

It is clear that the factors of the last formula, $A_{n-s_k}^{(1)}$, are exterior factors of the formula K_0' and, consequently, of the formula K_0. Since, however, all the factors of the formula K_0 are elementary regular, the factors of the formula $A_{n-s_k}^{(1)}$ are also elementary regular. It follows from this that the series of formulae

$$A_{n-s_k}^{(1)}, \ldots, A_n^{(1)}$$

is a regularity series. In virtue of this, the formula $A_n^{(1)}$—or, equivalently, $A^{(1)}$—is a regular formula.

2. *If all exterior factors of the formula A are regular, then the formula A is also regular.*

In fact, suppose $A_1, \ldots, A_n$ are the exterior factors of the formula A. We form their product:

$$A_1 \& A_2 \& \ldots \& A_k.$$

Since every formula A_i is regular, by application of the operations 1, 2, 3, it is reducible to a formula all exterior factors of which are elementary regular. We denote by A_i^* the product of all exterior factors of the formula A_i. By a successive application of the operations 1, 2, 3 to A_1, we can transform formula A into a formula in which the product of exterior factors has the form:

$$A_1^* \& A_2 \& \ldots \& A_n.$$

Applying, further, to this formula the sequence of operations 1, 2, 3, which transform the formula A_2, we obtain a formula in which the product of exterior factors has the form:

$$A_1^* \& A_2^* \& A_3 \& \ldots \& A_n.$$

Continuing this process, we finally obtain a formula in which the product of exterior factors has the form:

$$A_1^* \& A_2^* \& \ldots \& A_n^*.$$

Every factor of this formula is the product of elementary regular formulae. Consequently, the formula A is regular.

From properties 1 and 2, we obtain that:

3. *If a product is regular, then all the factors are regular; and, conversely, the product of regular factors is regular.*

4. *Every regular formula is deducible in restricted arithmetic.*

In fact, it is easily seen that operations 1, 2, 3 transform formulae in the predicate calculus into equivalent formulae. This follows from the validity of the following formulae in the predicate calculus:

$$A \,\&\, (B \vee (x)C(x)) \sim (x)(A \,\&\, (B \vee C(x)));$$

$$(\exists x)A(x) \sim A(y) \vee (\exists x)A(x);$$

$$A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n \vee B \sim (A_1 \vee B) \,\&\, (A_2 \vee B) \,\&\, \ldots \,\&\, (A_n \vee B).$$

These formulae are easily deduced in the predicate calculus. The first follows from Theorems 1, 2, §12, Chapter IV. The third is the distributive transformation. The second formula is also easily deduced. In fact, we obviously have that

$$\vdash (\exists x)A(x) \to A(y) \vee (\exists x)A(x).$$

In order to prove the reverse implication, we make a substitution in the following axiom of the propositional calculus:

$$(A \to C) \to ((B \to C) \to (A \vee B \to C)),$$

obtaining

$$\vdash (A(y) \to (\exists x)A(x)) \to (((\exists x)A(x) \to (\exists x)A(x)) \to$$
$$\to (A(y) \vee (\exists x)A(x) \to (\exists x)A(x))).$$

Both antecedents are true (the first is an axiom in the predicate calculus). Applying the law of deduction twice, we obtain that

$$\vdash A(y) \vee (\exists x)A(x) \to (\exists x)A(x).$$

Thus, both implications of the required equivalence are proved.

It follows from the validity of the first and third formulae that the operations 1 and 3 transform formulae of the predicate calculus into equivalent formulae. Replacing in the second formula the variable y by the arbitrary recursive term t, we obtain the following formula which is valid in restricted arithmetic:

$$(\exists x)A(x) \sim A(t) \vee (\exists x)A(x).$$

It follows from this that the operation 2 also transforms formulae of the predicate calculus into equivalent formulae.

An elementary regular formula is deducible in restricted arithmetic as it has the form $A \vee B$, where A is primitively valid and consequently it is, by definition, deducible in restricted arithmetic. But then the product of elementary regular formulae is also deducible in restricted arithmetic. The

introduction of exterior quantifiers also leads to a formula which is deducible in restricted arithmetic. Therefore K_0—which is the first formula in the regularity series—is deducible in restricted arithmetic. But then all the formulae $K_1, \ldots, K_n$ are also deducible in restricted arithmetic since they are obtained from K_0 by the successive application of elementary transformations. Consequently, every regular formula is deducible in restricted arithmetic.

5. If the regular formula A' is the result of the application of one of the operations 1, 2, 3 to the formula A, then formula A is also regular. This follows directly from the definition of a regular formula.

We have seen that the cancellation of exterior universal quantifiers does not disturb regularity.

6. The elimination of an arbitrary universal quantifier from a regular formula results in a regular formula.

Suppose the formula K_n' is obtained from the regular formula K_n by eliminating some universal quantifier. We shall consider the regularity series of the formula K_n:

$$K_0, K_1, \ldots, K_n.$$

We shall assume that the quantifier under consideration has been subjected to operation 1 in the formula K_p. After this, it becomes an exterior quantifier in the formula K_{p-1}. But if we eliminate it from all the formulae

$$K_{p-1}, K_p, \ldots, K_n$$

we obtain the series

$$K_{p-1}', K_p', \ldots, K_n'.$$

It is clear that the formulae $K_{n-1}', K_{n-2}', \ldots, K_p'$ are obtained from K_n' by the successive application of the operations 1, 2, 3. The formula K_p' coincides with the formula K_{p-1}' which is regular as it is obtained from the regular formula K_{p-1} by eliminating an exterior quantifier. In this case, the formula is transformed, by means of operations 1, 2, 3, into the regular formula K_p', which in turn is transformed by the same operations into a formula in which all exterior factors are elementary regular. It follows from this that the formula K_n' is regular.

In the case when the quantifier to be eliminated is not subjected to operation 1 in any of the formulae of the series

$$K_0, \ldots, K_n,$$

we cancel it in all formulae of the series. We obtain the series

$$K_0', \ldots, K_n',$$

which will also be a regularity series. In fact, K_n' is transformed into K_0' by means of the same operations which transform K_n into K_0; moreover, every exterior factor of K_0' has the same primitively valid sum of terms as the corresponding factor in K_0 since this summand, in which the cancelled quantifier occurs, could not occur in this primitively valid sum which, by definition, does not contain quantifiers.

§6. Some lemmas on regular formulae

LEMMA I. *If in a regular formula, certain summands of exterior factors are products, then after the cancellation of an arbitrary number (but not of all) factors in all of these products we obtain a regular formula.*

Suppose K is a regular formula and let

$$K_0, K_1, \ldots, K_n$$

be its regularity series, where K_n is K. We shall prove the lemma by applying induction on the length of this series. The lemma is true for K_0. In fact, all the exterior factors of K_0 are elementary regular. Every factor has the form $A \vee B$, where A is a primitively valid formula. The terms of this factor, being products, occur either in A or in B. After cancelling certain factors from its terms, the formula A goes over into the formula A', which is also primitively valid (this follows directly from the properties of the formulae of propositional algebra). In general, the terms occurring in B can be transformed arbitrarily since the composition of B does not have any influence on the problem of the elementary regularity of the formula $A \vee B$. Consequently, our assertion is true for K_0. We shall assume that it is true for K_{i-1} and show that it is then true for K_i also.

The formula K_{i-1} is obtained from K_i by the application of one of the operations 1, 2, 3 to one of the exterior factors of K. If one of the operations 1 (or 2) were applied, then an exterior factor of K_i of the form

$$(x)A(x) \vee B \text{ [or } (\exists x)A(x) \vee B]$$

would go over into a factor of K_{i-1} of the form

$$A(x) \vee B \text{ [or } A(t) \vee (\exists x)A(x) \vee B],$$

where t is an arbitrary recursive term. The remaining factors of K_i remain unchanged. We shall assume that we have cancelled such factors from the terms of the exterior factors of K_i. This cancellation can be effected either from the terms of the exterior factors which remain invariant in the transition to K_{i-1} or in the factor $(x)A(x) \vee B$ (or in $(\exists x)A(x) \vee B$ respectively) from the terms occurring in B, as $(x)A(x)$ and $(\exists x)A(x)$ are not products. Terms in the exterior factors of K_i which are products remain invariant in the transition to K_{i-1}. Cancelling certain factors from the terms of the exterior factors of K_i and the same factors from the terms of the exterior factors of K_{i-1}, we obtain the formulae K_i and K'_{i-1}. The first of these contains the exterior factor $(x)A(x) \vee B'$ (or $(\exists x)A(x) \vee B'$) respectively). The second has the same factors as the first with the exception of the one indicated which goes over into the factor

$$A(x) \vee B' \quad (\text{or } A(t) \vee (\exists x) \vee B')$$

(B' is the result of cancelling the factors in the terms of B). Therefore, the formula K'_{i-1} is the result of applying to the formula K'_n the same operation which transforms K_i into K_{i-1}. In virtue of the induction assumption, the

formula K'_{i-1} is regular. And then (in virtue of 5, §5) K_i is also regular. Thus, in the case when K_{i-1} is obtained from K_i by the operations 1 or 2, it follows from the regularity of K'_{i-1} that K'_i is regular.

We now assume that K_{i-1} is obtained from K_i by applying operation 3. In this case, one of the exterior factors of K_i, say

$$A_1 \,\&\, A_2 \,\&\, \dots \,\&\, A_k \vee B$$

goes over into the product of exterior factors of K_{i-1}:

$$(A_1 \vee B) \,\&\, (A_2 \vee B) \,\&\, \dots \,\&\, (A_k \vee B),$$

where $A_1, A_2, \dots, A_k$ are prime factors. The remaining factors of K_i remain unchanged in the transition to K_{i-1}. We shall assume that we have cancelled some of the factors from certain exterior terms of K_i which are products. In this connection, we could cancel some factors from the product

$$A_1 \,\&\, \dots \,\&\, A_k$$

also. Suppose these are the factors

$$A_1, A_2, \dots, A_p, p < k.$$

We shall denote by K'_i the formula obtained in this way. We cancel in K_{i-1} all of those exterior terms which do not change in the transition from K_i to K_{i-1}, the same factors which are cancelled in the formula K_i. Moreover, we eliminate from K_{i-1} the exterior factors:

$$A_1 \vee B', A_2 \vee B', \dots, A_p \vee B',$$

where B' is the result of cancelling in B. In virtue of 3, §5, the elimination of exterior factors does not disturb the regularity of a formula. We denote the formula obtained by K'_{i-1}. In virtue of the induction assumption, the formula K'_{i-1}, which was obtained from K_{i-1} by the cancellation of certain factors of exterior terms and the elimination of certain exterior factors, remains regular. But K'_{i-1} is obtained from K'_i by applying operation 3 and, consequently, K'_{i-1} is also regular. And so our assertion is valid for K_0 and from its validity for K_{i-1} follows its validity for K_i. Consequently, it is valid for an arbitrary regular formula—as was to be proved.

LEMMA II. *If arbitrary summands are joined to a summand of exterior factors of a regular formula, the formula remains regular.*

LEMMA III. *If a free object variable of a regular formula is replaced by an arbitrary term, then the formula remains regular.*

The proof of these two lemmas is easily carried out by induction exactly as in the proof of Lemma I.

REMARK 1. If we operate with one of the operations 1, 2, 3 on a regular formula, then we again obtain a regular formula. This assertion is not immediately obvious as such an operation might not be the one which trans-

forms the given formula into a formula of a regularity series. In order to prove our assertion, it suffices to show that the exterior factor, on which this operation is performed, remains regular. But if this operation is the operation 1, then the exterior factor considered goes over into one in which some universal quantifier is cancelled. On the basis of remark 6, §5, this factor remains regular. If operation 2 is applied to it, it goes over into a factor to which one more term is added [i.e. a factor of the form $(\exists x)A(x) \vee B$ goes over into a factor of the form $A(t) \vee (\exists x)A(x) \vee B$]. By virtue of Lemma II, the regularity is not disturbed. Finally, if operation 3 is performed, then a factor of the form

$$A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_k \vee B$$

goes over into a product of factors:

$$(A_1 \vee B) \,\&\, (A_2 \vee B) \,\&\, \ldots \,\&\, (A_k \vee B).$$

But each of the factors of this product can be obtained from the preceding one by cancelling all factors in the product $A_1 \,\&\, \ldots \,\&\, A_k$ except one. Therefore, by virtue of Lemma I, all the formulae $A_i \vee B$ are regular and consequently their product is also regular.

LEMMA IV. *If the formulae $A \vee K$ and $B \vee K$ are regular, then the formula $A \,\&\, B \vee K$ is also regular.*

Proof. We consider first the case when A and B are primitive formulae. By virtue of the regularity of the formula $A \vee K$, upon application of operations 1, 2, 3 it reduces to a formula K_0 in which all exterior factors are elementary regular. Furthermore, since A is a primitive formula, the operations performed never involve the term A (see p. 251). Therefore, all exterior factors of the formula K have the form

$$A \vee C' \vee K',$$

where $A \vee C'$ is a primitively true formula.

If we consider the formula in which A is replaced by an arbitrary formula X, i.e. $X \vee K$, then, performing the same operations on it as we performed on $A \vee K$, we obtain a formula whose exterior factors have the form:

$$X \vee C' \vee K'.$$

But these formulae are no longer, generally speaking, elementary regular. If we take the formula B as X, then the formulae

$$B \vee C' \vee K'$$

are regular, since they are exterior factors of the formula obtained by application of operations 1, 2, 3 to the formula $B \vee K$, which, by assumption, is regular. In this case, from every formula $B \vee C' \vee K'$ we can obtain with the aid of operations 1, 2, 3 a formula whose exterior factors are elementary regular formulae having the form

$$B \vee C' \vee C'' \vee K'',$$

where the formulae $\boldsymbol{B} \vee C' \vee C''$ are primitively true. Then also from the formula $X \vee C' \vee K'$, by the same operations, one obtains the formula

$$X \vee C' \vee C'' \vee K''.$$

We now replace X by the formula $A \,\&\, \boldsymbol{B}$. In this case, with our operations we first obtain the family of formulae

$$A \,\&\, \boldsymbol{B} \vee C' \vee K',$$

and then the family of formulae

$$A \,\&\, \boldsymbol{B} \vee C' \vee C'' \vee K''. \tag{1}$$

We shall show that every formula $A \,\&\, \boldsymbol{B} \vee C' \vee C''$ is primitively true. In fact, by means of identity transformations of propositional algebra, it can be transformed into the formula

$$(A \vee C' \vee C'') \,\&\, (\boldsymbol{B} \vee C' \vee C'').$$

Every factor of this formula is a primitively true formula. Therefore, the product and consequently also the formula

$$A \,\&\, \boldsymbol{B} \vee C' \vee C''$$

are primitively true. Consequently, all the formulae (1) are elementary regular and hence the formula $A \,\&\, \boldsymbol{B} \vee K$ is regular.

If at least one of the formulae A, $\boldsymbol{B}$ is not a primitive formula, the proof is simpler. We write A and $\boldsymbol{B}$ in the form of products:

$$A = A_1 \,\&\, \ldots \,\&\, A_p; \quad \boldsymbol{B} = \boldsymbol{B}_1 \,\&\, \ldots \,\&\, \boldsymbol{B}_q$$

(in particular, p and q may turn out to be equal to 1).

The formulae $A \vee K$, $\boldsymbol{B} \vee K$ and $A \,\&\, \boldsymbol{B} \vee K$ then assume the form:

$$A_1 \,\&\, \ldots \,\&\, A_p \vee K,\ \boldsymbol{B}_1 \,\&\, \ldots \,\&\, \boldsymbol{B}_q \vee K,$$

$$A_1 \,\&\, \ldots \,\&\, A_p \,\&\, \boldsymbol{B}_1 \,\&\, \ldots \,\&\, \boldsymbol{B}_q \vee K,$$

respectively.

The first two formulae are regular. Therefore, by virtue of Lemma I, each of the formulae $A_i \vee K$ and $\boldsymbol{B}_j \vee K$ is regular. Since the product $A \,\&\, \boldsymbol{B}$ is not a primitive formula, we can apply operation 3 to the formula

$$A_1 \,\&\, \ldots \,\&\, \boldsymbol{B}_q \vee K.$$

As a result of this, it goes over into the formula

$$(A_1 \vee K) \,\&\, \ldots \,\&\, (A_p \vee K) \,\&\, (\boldsymbol{B}_1 \vee K) \,\&\, \ldots \,\&\, (\boldsymbol{B}_q \vee K).$$

This formula is the product of regular factors and consequently it is regular. And then the formula $A \,\&\, \boldsymbol{B} \vee K$ is also regular.

COROLLARY. *Let the formula A be regular and suppose its exterior factors have the form $A' \vee \boldsymbol{B}' \vee \boldsymbol{B}''$ where $C \vee \boldsymbol{B}'$ is a regular formula. Then, replacing in the formula A all or some exterior factors by the formulae $A' \,\&\, C \vee \boldsymbol{B}' \vee \boldsymbol{B}''$ and renaming the variables (if this is necessary), we obtain a regular formula.*

REMARK 2. It follows from the lemmas just proved that *the application of distributive transformations to regular formulae also leads to regular formulae.* In particular, for the second distributive law this assertion can be formulated in the following manner:

If A_1 & ... & $A_n \vee B$ *is a regular formula, then* $(A_1 \vee B)$ & ... & $(A_n \vee B)$ *is also a regular formula, and conversely.* The proof of this assertion is quickly obtained from Lemmas I, IV and property 3 of regular formulae (see page 256). From this it follows that we can apply the indicated transformations also to the individual factors of a formula, which is a product, keeping the remaining factors unchanged. The application of the first distributive law is also possible. However, we shall not touch upon this here since we shall not need it.

LEMMA V. *If the formulae* $(\exists x)(A(x) \vee L)$ *and* $B \vee L$ *are regular, then the formula* $(\exists x)(A(x)\ \&\ B) \vee L$ *is also regular.*

Proof. Let $K_0, K_1, \ldots, K_n$ be a regularity series of the formula

$$(\exists x)A(x) \vee L.$$

Since, under all these operations, the term $(\exists x)A(x)$ does not disappear from the formula, every exterior factor of the formula K_i has the form:

$$(\exists x)A(x) \vee L^{(i)}.$$

For all exterior factors of the formula K_0 the assertion of the lemma is valid. In fact, if the formula $(\exists x)A(x) \vee L'$ is elementary regular, then L' contains primitively true terms. Therefore, whatever the formula X may be, the formula $X \vee L'$ is elementary regular. Consequently, in this case, the formula

$$(\exists x)(A(x)\ \&\ B) \vee L'$$

is also elementary regular.

We shall assume that our assertion is valid for all exterior factors of the formula K_{i-1} and prove that it is then valid for all exterior factors of K_i.

In fact, let

$$(\exists x)A(x) \vee L^{(i)}$$

be an exterior factor of the formula K_i and suppose the formula $B \vee L^{(i)}$ is regular. Among the exterior factors of K_{i-1}:

(1) either a factor

$$A(t) \vee (\exists x)A(x) \vee L^{(i)}$$

can be found;

(2) or a factor

$$(\exists x)A(x) \vee L^{(i-1)}$$

can be found, which is obtained as a result of applying one of the operations 1, 2, 3 to the formula $L^{(i)}$;

(3) or, finally, the factor

$$(\exists x)A(x) \vee L^{(i)}$$

itself is contained among the exterior factors of K_{i-1}.

In case (1), by virtue of the induction assumption and Lemma II, if $B \vee L^{(i)}$ is regular, then the formula

$$A(t) \vee (\exists x)(A(x) \,\&\, B) \vee L^{(i)}$$

is also regular. By virtue of the corollary to Lemma IV, then the formula

$$A(t) \,\&\, B \vee (\exists x)(A(x) \,\&\, B) \vee L^{(i)}$$

is regular. In this case, the formula

$$(\exists x)(A(x) \,\&\, B) \vee L^{(i)} \tag{2}$$

is regular, as the preceding formula is obtained from it by an application of operation 2.

In case (2), the operation performed on the formula does not apply to the term $(\exists x)A(x)$. It is performed on some member occurring in the composition of $L^{(i)}$. Therefore, this same operation on the formula $X \vee L^{(i)}$, where X is an arbitrary formula, leads to the formula $X \vee L^{(i-1)}$. This formula is regular if $X \vee L^{(i)}$ is regular, and conversely. Therefore, if the formula $B \vee L^{(i)}$ is regular, then $B \vee L^{(i-1)}$ is also regular. Then, by virtue of the induction assumption, the formula

$$(\exists x)(A(x) \,\&\, B) \vee L^{(i-1)} \tag{*}$$

is regular and hence also formula (2), from which (*) is obtained by one of the operations 1, 2, 3, is regular.

For the case (3), the induction is clear. We have thus proved that if the lemma is valid for the exterior factors of K_{i-1}, then it is also valid for the exterior factors of K_i. Consequently, it is valid for the formula $(\exists x)A(x) \vee L$, consisting of one exterior factor.

We note further one very simple fact which will be useful in the sequel.

REMARK 3. *If the formula $(\exists x)A(x) \vee L' \vee L''$ is regular, then it remains regular after the inclusion of the term L' under the sign of the quantifier $(\exists x)$.*

Let us consider the formula

$$(\exists x)(A(x) \vee L') \vee L''.$$

We perform operation 2 on it, replacing x by the term 0 in the isolated term:

$$A(0) \vee L' \vee (\exists x)(A(x) \vee L') \vee L''. \tag{3}$$

This formula contains among its terms all the terms of the formula

$$(\exists x)A(x) \vee L' \vee L'', \tag{4}$$

with the exception of $(\exists x)A(x)$, which is replaced by the term

$$(\exists x)(A(x) \vee L').$$

If, starting from formula (4), we obtain a set of formulae with the aid of

operations 1, 2, 3, then, starting with formula (3), we obtain—with the aid of the same operations—another set of formulae which differ from the first in that instead of the term $(\exists x)A(x)$ they contain the term $(\exists x)(A(x) \vee L')$ and, furthermore, in some exterior factors there can appear superfluous terms which do not occur in the formulae of the first set. But if we have thus reduced formula (4) to a formula in which all the exterior factors are elementary regular, then, obviously, the same operations also reduce formula (3) to a formula all exterior factors of which are elementary regular. It follows from this that if formula (4) is regular then formula (3) is also regular. And then the formula $(\exists x)(A(x) \vee L') \vee L''$, from which (3) is obtained by operation 2, is also regular.

We shall denote the reduced form of the formula $\bar{A}$ by A^-.

LEMMA VI. *The formula*

$$A(x_1, \ldots, x_n) \vee A^-(y_1, \ldots, y_n) \vee \overline{x_1 = y_1} \vee \ldots \vee \overline{x_n = y_n}$$

is regular whatever the formula A may be.

Proof. We shall give the proof of this theorem only for $n = 1$ since it is essentially the same in the general case.

We shall carry out the proof by induction on the structure of the reduced formula. We shall prove that the lemma holds if $A(x)$ is an elementary formula. In this case, the formula

$$A(x) \vee A^-(y) \vee \overline{x = y}$$

is primitively true.

In fact, if we replace x and y by numerals z_1 and z_2, we obtain the formula

$$A(z_1) \vee A^-(z_2) \vee \overline{z_1 = z_2}.$$

If the numerals are distinct, then the term $\overline{z_1 = z_2}$ has the value T and hence the entire formula also has the value T. If the numerals coincide, then the term

$$A(z) \vee A^-(z)$$

always has the value T.

We shall assume that our assertion is valid for the formulae A_1 and A_2; we shall show that it is then also valid for the formulae

$$A_1 \,\&\, A_2 \text{ and } A_1 \vee A_2.$$

In fact, by assumption, the formulae

$$A_1(x) \vee A_1^-(y) \vee \overline{x = y} \text{ and } A_2(x) \vee A_2^-(y) \vee \overline{x = y}$$

are regular; therefore, on the basis of Lemma II, the formulae

$$A_1(x) \vee A_1^-(y) \vee A_2^-(y) \vee \overline{x = y}$$

and

$$A_2(x) \vee A_2^-(y) \vee A_1^-(y) \vee \overline{x = y}$$

are also regular.

Then, in virtue of Lemma IV, the formula

$$A_1(x) \,\&\, A_2(x) \vee A_1^-(y) \vee A_2^-(y) \vee \overline{x = y}$$

is also regular. Since the formula

$$A_1^-(y) \vee A_2^-(y)$$

is the reduced form of the formula $\overline{A_1(y) \,\&\, A_2(y)}$, our lemma is proved for the formula $A_1 \,\&\, A_2$. The formula $A_1 \vee A_2$ is considered analogously. If the lemma holds for the formula A, then it is also valid for $\bar{A}$ since the expression

$$A(x) \vee A^-(y) \vee \overline{x = y}$$

is the same for both formulae.

We shall assume that the lemma is valid for $A(t, x)$; we shall show that it is also valid for the formulae

$$(t)A(t, x) \text{ and } (\exists t)A(t, x).$$

By assumption, the formula

$$A(t, x) \vee A^-(t, y) \vee \overline{x = y}$$

is regular. Obviously, $A(s, x) \vee A^-(s, y) \vee \overline{x = y}$ is also regular whatever the object variable s may be.

The formula

$$(t)A(t, x) \vee ((t)A(t, y))^- \vee \overline{x = y}$$

can be written in the form

$$(t)A(t, x) \vee (\exists t)A^-(t, y) \vee \overline{x = y}. \tag{5}$$

We first perform operation 1 on this formula:

$$(u)(A(u, x) \vee (\exists t)A^-(t, y) \vee \overline{x = y}),$$

and next operation 2:

$$(u)(A(u, x) \vee A^-(u, y) \vee (\exists t)A^-(t, y) \vee \overline{x = y}).$$

It follows from the regularity of the formula

$$A(u, x) \vee A^-(u, y) \vee \overline{x = y}$$

that the preceding formula is regular and consequently formula (5) is also regular.

The proof for the formula $(\exists t)A(t, x)$ is carried out analogously. We have thus proved our lemma for all reduced formulae.

REMARK 4. *The formula $A \vee A^-$ is always regular.*

We shall not prove this assertion since its proof is carried out by following the pattern of that of the preceding lemma.

LEMMA VII. *If one performs a substitution in the regular formula A, replacing the propositional variable A or the predicate variable $A(x, \ldots, u)$ by*

the formula B *or* $B(x, \ldots, u)$ *respectively, then a formula is obtained whose reduced form is regular.*

Proof. We shall carry out the proof for the case of a substitution in a predicate. The same line of reasoning with certain simplifications is suitable for the case of a substitution in a propositional variable.

Furthermore, we shall consider only a substitution in a predicate of one variable since the proof for a predicate of an arbitrary number of variables does not differ essentially from the case $n = 1$, except that one would have to write out more cumbersome expressions.

Suppose the regular formula A contains a predicate of one variable, $A(t)$. We shall show that the formula N, obtained from A by the replacement of $A(t)$ by the formula $B(t)$, is regular.

We consider first the case when A is a primitively true formula. Applying to it distributive transformations according to the second distributive law. we can reduce it to its conjunctive normal form A', which is also a primitively true formula. We take an arbitrary factor of this formula and isolate members in it which contain the predicate $A(\ )$. Then this factor can be represented in the form

$$A(x_1) \vee \ldots \vee A(x_p) \vee \bar{A}(y_1) \vee \ldots \vee \bar{A}(y_q) \vee L. \tag{6}$$

This factor, obviously, is also a primitively true formula. We shall prove that the formula

$$\Sigma(x_i = y_j) \vee L, \tag{7}$$

where the sign Σ denotes the logical sum of summands for all i from 1 to p and j from 1 to q, is primitively true. Formula (6) is primitively true; therefore, if we replace the variable predicate $A(\ )$ in it by an arbitrary recursive predicate, we obtain a formula which is also primitively true. We introduce the variables $s_1, s_2, \ldots s_p$, which do not occur in formula (6), and we perform a substitution, replacing the predicate $A(t)$ by the formula

$$\prod_i \overline{(s_i = t)},$$

where Π denotes the logical product of the factors for $i = 1, 2, \ldots, p$.

We obtain the primitively true formula

$$\prod_i \overline{(s_i = x_1)} \vee \prod_i \overline{(s_i = x_2)} \vee \ldots \\ \ldots \vee \prod_i \overline{(s_i = x_p)} \vee \overline{\prod_i \overline{(s_i = y_1)}} \vee \ldots \vee \overline{\prod_i \overline{(s_i = y_q)}} \vee L.$$

In this connection, L does not change since it does not contain the predicate A; transforming this formula according to the rules of propositional algebra, we obtain

$$\prod_i \overline{(s_i = x_1)} \vee \ldots \vee \prod_i \overline{(s_i = x_p)} \vee \\ \vee \sum_i (s_i = y_1) \vee \ldots \vee \sum_i (s_i = y_q) \vee L.$$

Replacing every variable s_i by the variable x_i, we obtain the primitively true formula

$$\prod_i \overline{(x_i = x_1)} \vee \ldots \vee \prod_i \overline{(x_i = x_q)} \vee$$
$$\vee \sum_i (x_i = y_1) \vee \ldots \vee \sum_i (x_i = y_q) \vee L.$$

Each of the terms $\prod_i \overline{(x_i = x_1)}, \ldots, \prod_i \overline{(x_i = x_p)}$ contains a primitively false factor: the first, the factor $\overline{x_1 = x_1}$; the second $\overline{x_2 = x_2}$, and so forth. Therefore, each of these terms is primitively false. According to the laws of propositional algebra, after the elimination of these terms the formula remains primitively true. The remaining formula coincides with formula (7). Transforming this formula according to the rules of propositional algebra, we obtain the primitively true formula

$$\prod_{i,j} \overline{(x_i = y_j)} \rightarrow L.$$

On the other hand, the substitution in the formulae

$$A(x_1) \vee \ldots \vee A(x_p) \vee \bar{A}(y_1) \vee \ldots \vee \bar{A}(y_q) \vee \overline{x_i = y_j},$$
$$i = 1, 2, \ldots, p; j = 1, 2, \ldots, q,$$

in which the predicate $A(t)$ is replaced by the formula $B(t)$, leads to the formulae

$$B(x_1) \vee \ldots \vee B(x_p) \vee \bar{B}(y_1) \vee \ldots \vee \bar{B}(y_q) \vee \overline{x_i = y_j}.$$

The reduced form of each of these formulae has the form:

$$B(x_1) \vee \ldots \vee B(x_p) \vee B^-(y_1) \vee \ldots \vee B^-(y_q) \vee \overline{x_i = y_j}. \tag{8}$$

It contains the term

$$B(x_i) \vee B^-(y_j) \vee \overline{x_i = y_j}.$$

By virtue of Lemma VI, this formula is regular. On the basis of Lemma II, formula (8) is also regular; finally, applying Lemma IV, we see that the formula

$$B(x_1) \vee \ldots \vee B(x_p) \vee B^-(y_1) \vee \ldots$$
$$\ldots \vee B^-(y_q) \vee \prod_{i,j} \overline{(x_i = y_j)} \tag{9}$$

is regular. But then the formula

$$B(x_1) \vee \ldots \vee B(x_p) \vee B^-(y_1) \vee \ldots \vee B^-(y_q) \vee L \tag{10}$$

is also regular. In fact, we consider the regularity series

$$K_0, K_1, \ldots, K_m$$

of formula (9). The term $\prod_{i,j} \overline{(x_i = y_j)}$ is primitive; therefore, it is not subject to the action of operations 1, 2, 3 and enters without change in the exterior factors of formula K_0, where one can consider it to be included in the

primitively true part. We write the primitively true part of an arbitrary exterior factor of K_0 in the form

$$H \vee \prod_{i,j} \overline{(x_i = y_j)}.$$

If we replace

$$\prod_{i,j} \overline{(x_i = y_j)}$$

by L in all the formulae K_i, then the primitively true part of each exterior factor of the formula K_0 takes on the form

$$H \vee L.$$

But, since the formula

$$\prod_{i,j} \overline{(x_i = y_j)} \to L$$

is primitively true, upon arbitrary replacements, every time that

$$\prod_{i,j} \overline{(x_i = y_j)}$$

takes on the value T, L also takes on the value T; from this it follows that every formula $H \vee L$ is also primitively true. It follows from what we said above that by replacing in formula (9) the term

$$\prod_{i,j} \overline{(x_i = y_j)}$$

by L we again obtain a regular formula.

On the other hand, this formula is the reduced form of the result of the substitution considered above of $B(t)$ in place of $A(t)$ in an arbitrary factor of the normal formula A'. Consequently, this substitution transforms the formula A' into a regular formula which we shall denote by N'. If we denote by N the formula obtained by this same substitution from formula A, then, evidently, N' is obtained from N and conversely N from N' by means of the same distributive operations as was formula A' from A and, conversely, A from A'. Since these distributive operations represent an application of the second distributive law, it follows from the regularity of N' that the formula N is regular. So, *a substitution in a predicate variable of a primitively true formula does not disturb regularity.*

For the case of a substitution of a primitively true formula in a propositional variable, the proof is carried out analogously with certain simplifications. Remark 4 plays the role of Lemma VI.

We now go on to the consideration of the general case. Suppose the formula A is regular and contains the predicate variable $A(\ )$. [For the sake of brevity, we shall again limit ourselves to consideration of a predicate of one variable.] Let

$$K_0, \ldots, K_n$$

be a regularity series of formula A. Each exterior factor of formula K_0 is elementary regular and it can therefore be written in the form

$$H \vee L,$$

where H is a primitively true formula. Replacing in $H \vee L$ the predicate $A(t)$ by an arbitrary formula $B(t)$, we obtain the formula $H' \vee L'$. On the basis of what we have already proved, H', obtained from H by the substitution under consideration, is a primitively true formula. By virtue of this, every exterior factor of the formula H_0 goes over, after replacement of the predicate $A(t)$ by the formula $B(t)$, into an elementary-regular formula.

If we denote by K_0' the formula which is the result of the substitution under consideration into formula K_0, then all the exterior factors of K_0' are elementary regular. We now replace $A(t)$ by the formula $B(t)$ in all the formulae of the regularity series. We obtain the series

$$K_0', \ldots, K_m'.$$

It is easily seen that K_{i-1}' is obtained from K_i' by the same operation 1, 2 or 3 as is formula K_{i-1} from K_i. Since the formula K_0' is regular, all the formulae of the given series are also regular. But the last formula K_m' is the result of substituting formula $B(t)$ for the predicate $A(t)$ in the formula K_m, which is at the same time formula A. Thus, substitution in a predicate variable does not disturb the regularity of the formula. The validity of this assertion for a substitution in a propositional variable is proved analogously.

§7. Operations dual to the operations 1, 2, 3

Similar to the way we previously considered an arbitrary formula in the form (α), we can write any formula in the dual form (β):

$$(\exists x_1) \ldots (\exists x_n)(A_{11} \mathbin{\&} \ldots \mathbin{\&} A_{1p_1} \vee \ldots \vee A_{k1} \mathbin{\&} \ldots \mathbin{\&} A_{kp_k}). \qquad (\beta)$$

The description of formula (β) is analogous to the description of formula (α), except that it is carried out in dual terms. The sum

$$A_{11} \mathbin{\&} \ldots \mathbin{\&} A_{1p_1} \vee \ldots \vee A_{k1} \mathbin{\&} \ldots \mathbin{\&} A_{kp_k}$$

consists of prime factors which we shall call *exterior summands* and each product $A_{i1} \mathbin{\&} \ldots \mathbin{\&} A_{ip_i}$ consists of prime factors which we shall call *exterior factors*. The quantifiers $(\exists x_i)$ are also called *exterior* quantifiers. In a particular case there may be no exterior quantifiers. It may turn out that in formula (β) there is only one exterior summand or that an exterior summand contains only one factor. Under such conditions, formula (β) can represent an arbitrary formula.

For formulae, given in the form (β), we can define operations, dual to operations 1, 2, 3, which we shall denote by 1*, 2*, 3*, respectively.

Operation 1 is the removal of an existential quantifier from an exterior factor of the form $(\exists x)B(x)$ and, if necessary, renaming the variable bound by this quantifier.*

EXAMPLE. $(\exists x)((y)A(y) \mathbin{\&} \boldsymbol{B} \vee (\exists y)\bar{A}(y) \mathbin{\&} \boldsymbol{C})$.

Applying operation 1* and renaming the corresponding variable, we obtain the formula

$$(\exists x)(\exists z)((y)A(y) \mathbin{\&} \boldsymbol{B} \vee \bar{A}(z) \mathbin{\&} \boldsymbol{C}).$$

Operation 2, dual to operation 2, is called isolation from the universal quantifier.* We shall not describe it in detail—we shall rather limit ourselves to an example.

EXAMPLE. $(x)(y)((z)A(z, t) \mathbin{\&} A(x, t) \vee \bar{A}(x, x) \mathbin{\&} \bar{A}(y, y))$.

Applying operation 2* to this formula and replacing z in the isolated term by the recursive term $x + y$, we obtain

$$(x)(y)(A(x + y, t) \mathbin{\&} (z)A(z, t) \mathbin{\&} A(x, t) \vee \bar{A}(x, x) \mathbin{\&} \bar{A}(y, y)).$$

Operation 3*, dual to operation 3, has already appeared in propositional algebra. There it was called the *first distributive operation.* We shall also retain this nomenclature for it here. When it is impossible to confuse it with the second distributive operation, we shall call it simply the distributive operation. Like operation 3, *it is applied only to non-primitive terms.* This operation can be applied if an exterior summand occurs in the formula, one factor of which summand is the sum

$$(A_1 \vee \ldots \vee A_n) \mathbin{\&} \boldsymbol{B},$$

where $A_1, \ldots, A_n$ are prime summands. In this case, operation 3* consists in replacing this summand by the sum

$$A_1 \mathbin{\&} \boldsymbol{B} \vee A_2 \mathbin{\&} \boldsymbol{B} \vee \ldots \vee A_n \mathbin{\&} \boldsymbol{B}.$$

EXAMPLE. Applying operation 3* to the formula

$$(\exists x)(F(z) \mathbin{\&} (A \vee G(z)) \vee \bar{F}(z) \mathbin{\&} \bar{G}(z)),$$

we obtain the formula

$$(\exists z)(F(z) \mathbin{\&} A \vee F(z) \mathbin{\&} G(z) \vee \bar{F}(z) \mathbin{\&} \bar{G}(z)).$$

Each of the operations 1*, 2*, 3* is always connected with some exterior summand. Sometimes, in order to indicate the operation more exactly, we shall say that it is applied to the given exterior summand.

§8. Properties of the operations 1*, 2*, 3*

We take as our immediate problem to show that if we apply one of the operations 1*, 2*, 3* to a regular formula, then the formula remains regular. Namely, we shall show that if we apply one of the operations 1*, 2*, 3* to the formula $\boldsymbol{A}$, which is part of the regular formula $\boldsymbol{A} \vee \boldsymbol{H}$, then the formula obtained, $\boldsymbol{A}' \vee \boldsymbol{H}$, remains regular.

LEMMA 1. *If $A \vee H$ is a regular formula and A' is the result of operation* 1* *on the formula A, then $A' \vee H$ is also a regular formula.*

We write A in the form (β):

$$(\exists x_1) \ldots (\exists x_n)((\exists y)A_0(y) \,\&\, B \vee C). \tag{1}$$

Suppose $(\exists y)$ is the quantifier which is removed in operation 1*. Then, in this notation, the content of the lemma is: if the formula

$$(\exists x_1) \ldots (\exists x_n)((\exists y)A_0(y) \,\&\, B \vee C) \vee H \tag{2}$$

is regular, then the formula

$$(\exists x_1) \ldots (\exists x_n)(\exists y)(A_0(y) \,\&\, B \vee C) \vee H \tag{3}$$

is also regular.

We consider first two special cases:

1. $n = 0$ in formula (2), i.e. quantifiers are absent.
2. Formula (2) is elementary regular.

In case 1, formula (2) has the form

$$(\exists y)A_0(y) \,\&\, B \vee C \vee H.$$

By assumption, this formula is regular; then, on the basis of Lemma 1, §6, the formulae

$$(\exists y)A_0(y) \vee C \vee H \quad \text{and} \quad B \vee C \vee H$$

are also regular. By virtue of Lemma V and Remark 3 in §6, the formula

$$(\exists y)(A_0(y) \,\&\, B \vee C) \vee H$$

is regular, which is what we were required to prove.

We now consider case 2: formula (2) is elementary regular. This formula, considered in the form (α), consists of a single exterior factor. By the definition of elementary regularity, it must have a primitively true summand. We can now omit consideration of the case $n = 0$. In the case when n is not equal to zero, term (1) must occur in F since it is not primitive. In this case, no matter how we may change it, the regularity of the formula is not disturbed.

We shall now prove our lemma by means of a two-fold induction. We shall first carry out induction on the number of quantifiers $(\exists x_i)$ of formula (2). For the initial case—when this number is zero—our theorem is proved. We shall assume that it is valid when this number is $n-1$. We shall prove its validity when the number of quantifiers is n. We assume formula (2) to be regular. In order to form regularity series, we must consider it in the form (α); then formula (2) reduces to a single exterior factor Let

$$K_0, \ldots, K_m$$

be the regularity series of formula (2). We shall succeed in proving the lemma if we establish its validity for every exterior factor of an arbitrary formula K_t having the form (2) with the number of quantifiers $(\exists x_j)$ equal to n.

We shall carry out this proof by a second induction with respect to the regularity series. Our lemma is valid for an arbitrary exterior factor of the formula K_0 inasmuch as these exterior factors are elementary regular, and we already considered this case above. We shall assume that our assertion is valid for every exterior factor of the form (2) of the formula K_{i-1}. We shall show that it then holds also for every exterior factor of the form (2) of formula K_i. We now consider an arbitrary exterior factor L_i of the formula K_i, having the form (2):

$$(\exists x_1) \ldots (\exists x_n)((\exists y)A_1(y) \,\&\, B_1 \vee C_1) \vee H_1$$

(where, by assumption, $n > 0$).

Obviously, the factor L_i is regular. It will be sufficient for us to prove that the formula L_{i-1}, obtained from L_i by the application of operation 1* to the summand

$$(\exists x_1) \ldots (\exists x_n)((\exists y)A_1(y) \,\&\, B_1 \vee C_1), \tag{s}$$

is also regular.

We shall first consider the case when the operation carrying K_i into K_{i-1} does not apply to the summand (s). If this operation is not performed on the factor L_i, then everything is clear. Consequently, we can assume that it is performed on a factor L_i and carries it into the factor L_{i-1} of the formula K_{i-1}. If this is operation 3, then it carries L_i not into one factor L_{i-1} but into the product of several such factors. In every case, each factor L_{i-1} of the formula K_{i-1}, obtained from L_i upon the transition from K_i to K_{i-1}, is regular and contains the summand (s). Then, by the induction assumption, the formula L'_{i-1} obtained from L_{i-1} by applying operation 1* to its summand (s) will also be regular. Since the formula L'_{i-1} (in the case of operation 3, the product of the corresponding L'_{i-1}) is obtained from L'_i from the same operation 1, 2 or 3 as is L_{i-1} (in the case of operation 3, the product of the corresponding factors L_{i-1}) from L_i, then the regularity of L'_{i-1}, by virtue of property 5 of regular formulae (see §6), implies the regularity of L'_i.

We now proceed to the case when operation 1, 2 or 3 is applied to the summand (s) of the exterior factor under consideration. Then, obviously, this can only be operation 2, and an exterior factor goes over into an exterior factor of formula K_{i-1} of the following form:

$$(\exists x_2) \ldots (\exists x_n)((\exists y)A_1(y, t) \,\&\, B_1(t) \vee C(t)) \vee \\ \vee (\exists x_1) \ldots (\exists x_n)((\exists y)A_1(y) \,\&\, B_1 \vee C_1) \vee H_1.$$

By virtue of the second induction assumption, we can—without disturbing regularity—apply operation 1* to an arbitrary exterior factor of the formula K_{i-1} of the corresponding type. Therefore, the formula

$$(\exists x_2) \ldots (\exists x_n)((\exists y)A_1(y, t) \,\&\, B_1(t) \vee C_1(t)) \vee \\ \vee (\exists x_1) \ldots (\exists x_n)(\exists y)(A_1(y) \,\&\, B_1 \vee C_1) \vee H_1$$

is regular. By virtue of the first induction assumption, our lemma is valid

for the case when the number of quantifiers in formula (2) is $n-1$. We can apply operation 1* a second time to the formula obtained, only with respect to its first term. We obtain the regular formula

$$(\exists x_2) \ldots (\exists x_n)(\exists y)(A_1(y, t) \,\&\, B_1(t) \vee C_1(t)) \vee \\ \vee (\exists x_1) \ldots (\exists y)(A_1(y) \,\&\, B_1 \vee C_1) \vee H_1.$$

We consider the formula

$$(\exists x_1) \ldots (\exists x_n)(\exists y)(A_1(y) \,\&\, B_1 \vee C_1) \vee H_1. \tag{*}$$

The preceding regular formula is obtained from (*), above, by operation 2. Therefore, this formula is also regular. On the other hand, it is obtained from the exterior factor L_i of the formula K_i by applying operation 1* to the summand (s). Thus, if the lemma is valid for exterior factors of the form (2) of formula K_{i-1}, then it is also valid for exterior factors of the form (2) of formula K_i. Consequently, it is also valid for formula (2).

LEMMA II. *If the formula $A \vee H$ is regular and A' is the result of applying operation 2* to the formula A, then the formula $A' \vee H$ is also regular.*

We assume that formula A is given in the form (β). Then Lemma II can be formulated as follows: *Suppose the formula $A \vee H$ is regular and that some exterior summand of the formula A has the form $(z)A_0(z) \,\&\, B$. Then, replacing it in this formula by the summand $A_0(t) \,\&\, (z)A_0(z) \,\&\, B$, where t is an arbitrary recursive term, which does not lead to a collision of variables, we obtain the formula A', such that the formula $A' \vee H$ is regular.*

Formula A has the following form in the form (β):

$$(\exists x_1) \ldots (\exists x_n)((z)A_0(z) \,\&\, B \vee H). \tag{4}$$

We first consider two cases:

1. $n = 0$ in formula (4).
2. The formula $A \vee H$ is elementary regular.

In the first case, the formula $A \vee H$ has the form:

$$(z)A_0(z) \,\&\, B \vee C \vee H. \tag{5}$$

If it is regular, then, on the basis of Lemma I, the formulae

$$(z)A_0(z) \vee C \vee H \quad \text{and} \quad B \vee C \vee H$$

are also regular. Then the formula $A_0(z) \vee C \vee H$ is regular (see page 257). Further, on the basis of Lemma III, §6, the formula

$$A_0(t) \vee C \vee H$$

is regular. On the basis of Lemma IV, §6, the formula

$$A_0(t) \,\&\, (z)A_0(z) \,\&\, B \vee C \vee H$$

is also regular. But this formula is A', which is obtained from A by means of operation 2*.

In case 2, by virtue of elementary regularity, the formula $A \vee H$ contains

a primitively true part in which the summand $(z)A_0(z) \& B$ cannot occur. Therefore, this summand can be replaced by any other one without disturbing regularity.

After this, we carry out the proof of our lemma by means of a two-fold induction. Firstly, we shall carry out induction on the number of quantifiers $(\exists x_i)$ of the formula (4), i.e. on the number n. For $n = 0$, the validity of the lemma has already been established. We shall assume that the lemma is valid if the number of quantifiers is $n-1$ and we shall prove that it is then valid for n.

In this case, the formula $A \vee H$ has the form:

$$(\exists x_1) \ldots (\exists x_n)((z)A_0(z) \& B \vee C) \vee H. \tag{6}$$

By assumption, this formula is regular. Let

$$K_0, \ldots, K_m$$

be its regularity series. We shall prove that our lemma is valid for every exterior factor of an arbitrary formula K_i, having the form (6). By the same token, it will also be proved for the formula (6) itself, which consists of a single exterior factor. We shall carry out the proof by a second induction (from K_{i-1} to K_i).

As we saw, our lemma is valid for the formula K_0 (see case 2). We shall assume that it is valid for K_{i-1}.

We consider an arbitrary exterior factor of the formula K_i, having the form (6):

$$(\exists x_1) \ldots (\exists x_n)((z)A_1(z) \& B_1 \vee C_1) \vee H. \tag{7}$$

If K_{i-1} is obtained from K_i by an operation which is not performed on the term

$$(\exists x_1) \ldots (\exists x_n)((z)A_1(z) \& B_1 \vee C_1), \tag{8}$$

then this term goes over into K_{i-1} without change. Performing operation 2* on it in the formulae K_i and K_{i-1}, we obtain the formulae K_i' and K_{i-1}'. By the induction assumption, K_{i-1}' is a regular formula. But it is obtained from K_i' by the same operation 1, 2 or 3 as was the formula K_{i-1} from K_i. Consequently, K_i' is regular.

We shall assume that K_{i-1} is obtained from K_i by the operation performed on the term (8). This can then only be operation 2. In this case, the exterior factor K_i, containing the term (8), goes over into an exterior factor of K_{i-1} of the following form:

$$(\exists x_2) \ldots (\exists x_n)((z)A_1(z, t) \& B_1(t) \vee C_1(t)) \vee \\ \vee (\exists x_1) \ldots (\exists x_n)((z)A_1(z) \& B_1 \vee C_1) \vee H_1.$$

By the induction assumption, pertaining to the formula K_{i-1}, we can apply operation 2* to the second term without disturbing the regularity of the last formula. Then the formula

$$(\exists x_2) \ldots (\exists x_n)((z)A_1(z, t) \& B_1(t) \vee C_1(t_1)) \vee \\ \vee (\exists x_1) \ldots (\exists x_n)(A_1(g) \& (z)A_1(z) \& B_1 \vee C_1) \vee H_1$$

is regular. By the first induction assumption, pertaining to the number of quantifiers $(\exists x_i)$, we can apply operation 2* to the first term also since in it the number of quantifiers is equal to $n-1$. After this, we obtain the regular formula

$$(\exists x_2) \dots (\exists x_n)(A_1(g, t) \,\&\, (z)A_1(z, t) \,\&\, B_1(t) \vee C_1(t)) \vee \\ \vee (\exists x_1) \dots (\exists x_n)(A_1(g) \,\&\, (z)A_1(z) \,\&\, B_1 \vee C_1) \vee H_1.$$

This formula is the result of operation 2 on the formula

$$(\exists x_1) \dots (\exists x_n)((z)A_1(z) \,\&\, A_1(g) \,\&\, B_1 \vee C_1) \vee H_1.$$

Consequently, the last formula is also regular. So, we have proved by induction from C_{i-1} to C_i that formula (6) is regular. We have thus finished the induction from $n-1$ to n and the lemma is completely proved.

LEMMA III. *Suppose A' is the result of operation* 3* *applied to the formula A. If the formula $A \vee H$ is regular, then the formula $A' \vee H$ is also regular.*

The proof of this lemma is carried out following the same pattern as the proof of the preceding two lemmas. Therefore, we limit ourselves to giving hints for it. We write A in the form (β), i.e. in the form

$$(\exists x_1) \dots (\exists x_n)((A_1 \vee \dots \vee A_p) \,\&\, B \vee C),$$

where $(A_1 \vee \dots \vee A_p) \,\&\, B$ is the term to which operation 3* is applied. In this case, the lemma can be expressed as follows:

If the formula

$$(\exists x_1) \dots (\exists x_n)((A_1 \vee \dots \vee A_p) \,\&\, B \vee C) \vee H \tag{9}$$

is regular, then the formula

$$(\exists x_1) \dots (\exists x_n)(A_1 \,\&\, B \vee \dots \vee A_p \,\&\, B \vee C) \vee H$$

is also regular.

One must first prove that the lemma is valid if $n = 0$ in formula (9) or if this formula is elementary regular. Furthermore, it is necessary to perform a two-fold induction, assuming that our lemma is valid when the number of quantifiers equals n, and, having proved its validity for $n + 1$, resorting to the second induction, applied to the regularity series.

LEMMA IV. *If the formula*

$$(\exists x_1) \dots (\exists x_n)(A_1 \,\&\, B_1 \vee \dots \vee A_p \,\&\, B_p) \vee H \tag{10}$$

is regular and every formula A_i is primitively false, then the formula H is regular.

We also include the case when $n = 0$ in the content of this lemma. But the case when H is absent is no longer possible here. Speaking more precisely, we assume that in the content of the lemma there also occurs the assertion that H cannot be absent or, equivalently, the formula

$$(\exists x_1) \dots (\exists x_n)(A_1 \,\&\, B_1 \vee \dots \vee A_p \,\&\, B_p) \tag{11}$$

cannot be regular. We assume further that the factor B_i cannot be omitted from the term $A_i \& B_i$.

We shall first prove that in an arbitrary regular sum we can eliminate an arbitrary false summand without disturbing regularity.

We consider the regularity series

$$K_0, \ldots, K_m$$

of the given formula, and prove this assertion for all exterior factors of the formulae K_i. It is valid for exterior factors of the formula K_0. In fact, let A^0 be a primitively false summand of some exterior factor of the formula K_0. If the summand A^0 does not occur in a primitively true part of this factor, then it can be eliminated. But if it occurs in a primitively true part, then this part can be written in the form

$$A^0 \vee G.$$

But it follows from propositional algebra that G must always receive the value T for all possible replacements in this formula as A^0 always receives the value F. Thus, G is itself primitively true and A^0 can be eliminated without disturbing the regularity of the exterior factor under consideration. We shall assume that our assertion is valid for the formula K_{i-1} and we shall show that it also holds for K_i. But a primitively false summand of an arbitrary exterior factor of K_i is not subject to the action of any of the operations 1, 2, 3, as is the case for every primitive formula. Therefore, it remains invariant in the formula K_{i-1} also. If we cancel it from the formulae K_i and K_{i-1}, we obtain the formulae K_i' and K_{i-1}' where K_{i-1}' is obtained from K_i' by the same operation 1, 2 or 3 as was K_{i-1} from K_i; but, by assumption, K_{i-1}' remains regular and consequently K_i' also. It follows immediately from what we have proved that our lemma is valid when there are no quantifiers $(\exists x_i)$ in the term (10), i.e. when $n = 0$. In this case, formula (9) has the form:

$$A_1 \& B_1 \vee \ldots \vee A_p \& B_p \vee H.$$

It follows from the regularity of this formula, on the basis of Lemma I, §6, that the formula

$$A_1 \vee A_2 \vee \ldots \vee A_p \vee H$$

is regular. But, in virtue of what has been proved, all primitively false terms A_i can be successively eliminated, without disturbing the regularity of the formula. Thus the formula H will also be regular.

So, our lemma is valid for $n = 0$. This done, it is easy to prove the lemma with the aid of a two-fold induction in the same way that the preceding lemmas were proved. We shall not carry out this line of reasoning—we shall only remark that when considering operation 2 one must make use of the assertion that if the formula $A(x)$ is primitively false then the formula $A(t)$, where t is an arbitrary recursive term, is also primitively false. The truth of this assertion, however, follows directly from the definition of primitive falsity.

§9. The regularity of formulae deducible in arithmetic

In the sequel, we shall call every formula of arithmetic whose reduced form is regular a *regular formula.*

THEOREM 1. *Every formula of arithmetic which is deducible from regular formulae by means of the rules of inference of arithmetic is regular.*

Proof. We recall that by definition the rules of inference of arithmetic coincide with the rules of the extended predicate calculus, which in turn differ from the rules of the predicate calculus only in that in connection with the introduction of terms the rules of substitution are extended. Consequently, it is sufficient for us to prove the assertion of the theorem for all rules of the extended predicate calculus.

1. *Rule for substitution in a propositional or predicate variable.* The validity of the theorem for this rule follows directly from Lemma VII, §6.

2. *Rule for substitution in an object variable.* This rule is obvious inasmuch as the structure of a formula does not change upon such a substitution.

3. *First rule for binding by a quantifier.* It is necessary to show that if the formula

$$\boldsymbol{B} \rightarrow A(x), \tag{1}$$

where $\boldsymbol{B}$ does not contain the variable x, is regular, then the formula

$$\boldsymbol{B} \rightarrow (x)A(x) \tag{2}$$

is also regular. We consider the reduced form of formula (1). It has the form (see page 264):

$$\boldsymbol{B}^{-} \vee A(x). \tag{3}$$

By assumption, this formula is regular. The reduced form of formula (2) has the form:

$$\boldsymbol{B}^{-} \vee (x)A(x). \tag{4}$$

Performing operation 1 on it (i.e. the operation of bringing out a universal quantifier), we obtain the formula

$$(x)(\boldsymbol{B}^{-} \vee A(x)).$$

Since the formula after the quantifier sign is regular, the entire formula is regular. But then formula (4) also and, consequently, formula (2) are regular.

4. *Second rule for binding by a quantifier.* This rule is considered in an analogous manner.

5. *Rule of inference.* Suppose the formulae A and $A \rightarrow \boldsymbol{B}$ are regular. It is required to show that the formula $\boldsymbol{B}$ is also regular. We write formula A (or its reduced form) in the form (α):

$$(x_1) \ldots (x_n)(A_{11} \vee \ldots \vee A_{1p_1}) \,\&\, \ldots \,\&\, (A_{m1} \vee \ldots \vee A_{mp_m}). \tag{5}$$

By assumption, this formula is regular. Consequently, by means of operations 1, 2, 3 it can be reduced to the form

$$(x_1) \dots (x_n)(y_1) \dots (y_q)(A_0^{(1)} \vee H_0^{(1)}) \,\&\, \dots \,\&\, (A_0^{(q)} \vee H_0^{(q)}), \qquad (6)$$

where all the $A_0^{(i)}$ are primitively true formulae. The reduced form of the formula $A \rightarrow B$ has the form:

$$A^- \vee B',$$

where A^- is the reduced form of $\bar{A}$ and B' is the reduced form of B. The formula A^- has the form

$$(\exists x_1) \dots (\exists x_n)(A_{11}^- \,\&\, \dots \,\&\, A_{1p_1}^- \vee \dots \vee A_{m1}^- \,\&\, \dots \,\&\, A_{mp_m}^-). \qquad (7)$$

This is, A^- written in the form (β) which is dual to the form (α). If we apply the operations 1*, 2*, 3*, which are dual to those which were applied to formula (5), to this formula, then we will obviously obtain the formula

$$(\exists x_1) \dots (\exists x_n)(A_0^{(1)-} \,\&\, H_0^{(1)-} \vee \dots \vee A_0^{(q)-} \,\&\, H_0^{(q)-}).$$

By virtue of Lemmas I, II, III which were proved in §8, the application of operations 1*, 2*, 3* to the term A^- in the formula $A^- \vee B'$ does not disturb the regularity of this formula. It follows from this that the formula

$$(\exists x_1) \dots (\exists x_n)(A_0^{(1)-} \,\&\, H_{(0)}^{(1)-} \vee \dots \vee A_0^{(q)-} \,\&\, H_0^{(q)-}) \vee B'$$

is regular. All the formulae $A_0^{(i)-}$ in this formula are primitively false inasmuch as the $A_0^{(i)}$ are primitively true. But then, by virtue of Lemma IV, §8, formula B', and consequently B also, is regular.

We have thus shown that the application of all the rules of inference of the extended predicate calculus to regular formulae leads to regular formulae, which is what we required to prove.

THEOREM 2. *Formulae which are deducible in restricted arithmetic are regular.*

Proof. To prove this theorem, it is now sufficient for us to establish that all the axioms of restricted arithmetic are regular. By virtue of Theorem 1, it easily follows from this that all formulae which are deducible in restricted arithmetic are regular. We shall consider first general logical axioms. They consist of the axioms of the propositional calculus and the two axioms of the predicate calculus.

The axioms of the propositional calculus and, consequently, their reduced forms are identically true formulae of the propositional calculus. Therefore, they are all primitively true and, consequently, regular.

The reduced form of the first axiom of the predicate calculus has the form

$$(\exists x)\bar{A}(x) \vee A(y).$$

Performing operation 2 on this formula, we obtain

$$\bar{A}(y) \vee (\exists x)\bar{A}(x) \vee A(y).$$

This formula is elementary regular; consequently, the axiom

$$(x)A(x) \to A(y)$$

is a regular formula.

The regularity of the second axiom of the predicate calculus is established in the same way.

We showed in §3 that all the axioms VI and VII of restricted arithmetic are primitively true formulae. Consequently, they are also regular.

Finally, in restricted arithmetic there appear initially true formulae of the form

$$f(x_1, \ldots, x_{n-1}, 0) = k(x_1, \ldots, x_{n-1}),$$

$$f(x_1, \ldots, x_{n-1}, x_n') = \xi(x_1, \ldots, x_n, f(x_1, \ldots, x_n)),$$

where

$$f(x_1, \ldots, x_{n-1}, 0),\ k(x_1, \ldots, x_{n-1}) \text{ and } \xi(x_1, \ldots, x_n, x_{n+1})$$

are recursive terms. The numerical interpretation of these equations is that we assign to the formula

$$f(0^{(k_1)}, \ldots, 0^{(k_{n-1})})$$

[to the formula

$$f(0^{(k_1)}, \ldots, 0^{(k_{n-1})}, 0^{(k_n+1)})]$$

the same numeral which is assigned to the formula

$$k(0^{(k_1)}, \ldots, 0^{(k_{n-1})})$$

[to the formula

$$\xi(0^{(k_1)}, \ldots, 0^{(k_n)}, f(0^{(k_1)}, \ldots, 0^{(k_n)})),$$

respectively]. Therefore, all formulae of this type are primitively true and, consequently, regular. Thus, all the axioms of restricted arithmetic are regular. Consequently, by virtue of Theorem 1, all formulae which are deducible in restricted arithmetic are regular, which is what we required to prove.

We remark that the regularity of a formula implies its weak regularity. Thus, all formulae which are deducible in restricted arithmetic are also weakly regular.

§10. Consistency of restricted arithmetic

THEOREM. *Restricted arithmetic is consistent.*

Proof. As we have already pointed out more than once, the problem of the consistency of any of the calculi we are considering is equivalent to the problem of the existence in it of a non-deducible formula. For the proof of consistency it is sufficient in this case to prove the existence in the calculus under consideration of a non-deducible formula. We shall prove that the formula $\overline{0 = 0}$ is not deducible in restricted arithmetic. In fact, if this formula were deducible in restricted arithmetic, then, by virtue of the remark we made at the end of §9, it would be a weakly regular formula. But since

this formula is primitive, it must then be primitively true in the weak sense. However, it is not, since, by assumption, we must replace the proposition $0 = 0$ by the symbol T and, consequently, the proposition $\overline{0 = 0}$ has the value F. So, the formula $\overline{0 = 0}$, not being weakly regular, cannot be deducible in restricted arithmetic.

The significance of the result just obtained is that the consistency of the use of infinity within the bounds of restricted arithmetic is proved by formal and finitary means. We remark that all the calculi whose consistency we proved hitherto could make no use of infinity since the interpretations were made on finite fields. Restricted arithmetic cannot be interpreted in any finite field of objects. The method we described in this chapter allows us, as we shall see, to establish other facts which are not self-evident.

§11. Independence of the axiom of complete induction in arithmetic

In the following section we shall prove a stronger theorem on the independence of the axiom of complete induction which includes as a special case the theorem on the independence of the axiom of complete induction from the remaining axioms of arithmetic. We shall, however, first prove the theorem on the independence of the axiom of complete induction in arithmetic since, although it is weaker than the theorem which will be proved later, the proof of it is significantly simpler.

THEOREM. *The axiom of complete induction is not deducible from the remaining axioms of arithmetic.*

Proof. If the axiom of complete induction were deducible from the remaining axioms, then it would be weakly regular. We shall show that it cannot be weakly regular. The axiom of complete induction has the form

$$A(0) \,\&\, (x)(A(x) \to A(x')) \to A(y),$$

and its reduced form is

$$\bar{A}(0) \vee (\exists x)(A(x) \,\&\, \bar{A}(x')) \vee A(y). \tag{1}$$

We shall assume that it is weakly regular. In this case, with the aid of operations 1, 2, 3, it can be reduced to a formula all exterior factors of which are elementary regular in the weak sense. However, the only operation which can be applied to formula (1) is the operation of isolation from the quantifier $(\exists x)$. Formulae obtained from (1) with the aid of this operation again allow the application of only this same operation, and so on. Every formula obtained from (1) by the application of operation 2 n times has the form

$$\bar{A}(0) \vee A(t_1) \,\&\, \bar{A}(t_1') \vee \ldots$$
$$\ldots \vee A(t_n) \,\&\, \bar{A}(t_n') \vee (\exists x)(A(x) \,\&\, \bar{A}(x')) \vee A(y). \tag{2}$$

Formula (2), as well as formula (1), coincides with its single exterior factor.

Therefore, if formula (1) is weakly regular, then, for some n, formula (2) is elementary regular in the weak sense. The formula

$$\bar{A}(0) \vee A(t_1) \,\&\, \bar{A}(t_1') \vee \ldots \vee A(t_n) \,\&\, \bar{A}(t_n') \vee A(y) \tag{3}$$

must therefore be primitively true in the weak sense. We replace the variable y by the numeral $0^{(n+1)}$ and the variables occurring in t_i (if there are any) by arbitrary numerals in formula (3). In this case, all the terms receive definite numerical values. Suppose the term t_i receives the value z_i. After the replacement of all terms by numerals, formula (3) takes on the form

$$\bar{A}(0) \vee A(z_1) \,\&\, \bar{A}(z_1') \vee \ldots \vee A(z_n) \,\&\, \bar{A}(z_n') \vee A(0^{(n+1)}). \tag{4}$$

We can assume that all the numerals z_i are distinct. If two numerals z_p and z_q did turn out to be equal, then there would be two identical summands $A(z_p) \,\&\, \bar{A}(z_p')$ and $A(z_q) \,\&\, \bar{A}(z_q')$ in formula (4). But after cancelling one of two identical summands (in a formula of the propositional calculus) we obtain a formula which is equivalent to the former.

Formula (4)—as a formula of propositional algebra—must take on the value T for all values of the logical variables occurring in it:

$$\bar{A}(0),\ A(z_1),\ A(z_1'),\ \ldots,\ A(z_n),\ A(z_n'),\ A(0^{(n+1)}).$$

In other words, this formula, considered as a formula of propositional algebra, must be identically true. If this were the case, then the formula

$$\bar{A}(0) \vee A(z_1) \vee \ldots \vee A(z_n) \vee A(0^{(n+1)}),$$

which is a factor of the conjunctive normal form of formula (4), would also be identically true. But, as is known, this can be the case only when some logical variable and its negation occur as summands in the formula. And this can occur only if one of the numerals z_i equals 0. Suppose, for example, that $z_i = 0$. We consider another factor of the conjunctive normal form of formula (4):

$$\bar{A}(0) \vee \bar{A}(0') \vee A(z_2) \vee \ldots \vee A(z_n) \vee A(0^{(n+1)}).$$

This formula must also be identically true. Therefore, one of the numerals $z_2, \ldots, z_n$ must be $0'$. Suppose this is z_2. Reasoning the same way further, we show that the numerals $z_1, \ldots, z_n$ must be respectively $0, 0', \ldots, 0^{(n-1)}$. Finally, we consider the following factor of the conjunctive normal form of formula (4):

$$\bar{A}(0) \vee \bar{A}(0') \vee \ldots \vee \bar{A}(0)^{(n-1)}) \vee A(0^{(n+1)}).$$

This factor can in no case be an identically true formula. Consequently, formula (4), and hence formula (3), cannot be primitively true in the weak sense. Hence, formula (1), i.e. the axiom of complete induction, cannot be weakly regular. But then, by virtue of Theorem 2, §9, the axiom of complete induction cannot be deduced from the remaining axioms of arithmetic, which is what we required to prove.

In the proof we have just given of the independence of the axiom of

complete induction from the remaining axioms of arithmetic, we observe that the assertion about the independence of this axiom can be strengthened and generalized to the case when the calculus contains other terms besides the recursive terms. All our reasoning, in every case, remains in force if we include in arithmetic arbitrary terms, connecting them in an arbitrary fashion by the relations $=$ and $<$, i.e. introducing for them auxiliary axioms of the form $\boldsymbol{t} = \boldsymbol{g}$ and $\boldsymbol{t} < \boldsymbol{g}$. In this connection, we shall assume that to every term, upon an arbitrary replacement of its variables by numerals, we can set into correspondence a definite numeral so that all the newly introduced axioms are satisfied. In this case, the new axioms are primitively true in the weak sense and, consequently, all the formulae which are deducible in the new calculus are weakly regular. But the proof of the fact that the axiom of complete induction cannot be weakly regular is in no way connected with the nature of terms and hence it is valid for the new calculus.

§12. Strengthened theorem on the independence of the axiom of complete induction

REMARK. The deduction theorem which we formulated in §4, Chapter V for the extended predicate calculus is valid for restricted arithmetic also. In the sequel, we shall use this theorem in the following form.

If formula $\boldsymbol{B}$ *is deducible from formula* $\boldsymbol{A}$ *in restricted arithmetic and if in this deduction no substitutions are made in the free object variables and predicate variables, occurring in formula* $\boldsymbol{A}$, *and if none of these variables are bound in the course of the deduction, then the formula* $\boldsymbol{A} \rightarrow \boldsymbol{B}$ *is deducible in restricted arithmetic.*

In fact, if formula $\boldsymbol{B}$ is deducible from $\boldsymbol{A}$ in restricted arithmetic, then this means that it is deducible from the axioms of the extended predicate calculus, and the axioms of arithmetic proper, to which we adjoin recursive equations, and formula $\boldsymbol{A}$. In this connection, the axioms of arithmetic proper can be replaced by formulae in which all object variables are bound. To this end, it is sufficient to bind all object variables by universal quantifiers. We retain for the formulae so obtained the name "axioms of arithmetic proper" (these formulae too do not contain propositional and predicate variables as the arithmetic axioms themselves from which they are formed do not contain them). Suppose

$$\boldsymbol{A}_1, \ldots, \boldsymbol{A}_p$$

are the axioms of arithmetic proper which are required for the deduction of formula $\boldsymbol{B}$. In this case, formula $\boldsymbol{B}$ is deducible from the formula

$$\boldsymbol{A} \,\&\, \boldsymbol{A}_1 \,\&\, \ldots \,\&\, \boldsymbol{A}_p$$

by means of the extended predicate calculus. In the deduction of formula $\boldsymbol{B}$, substitution in free variables in the last formula and binding of object variables by a quantifier can be avoided since they were not engendered by the variables

of formula A, and the formulae $A_1, \ldots, A_p$ do not contain free variables. Thus, the conditions of the deduction theorem are satisfied, and we conclude that the formula

$$A \mathbin{\&} A_1 \mathbin{\&} \ldots \mathbin{\&} A_p \rightarrow B$$

is deducible in the extended predicate calculus. In this case, the equivalent formula

$$A_1 \mathbin{\&} \ldots \mathbin{\&} A_p \rightarrow (A \rightarrow B)$$

is deducible in the extended predicate calculus. But the formula

$$A_1 \mathbin{\&} \ldots \mathbin{\&} A_p$$

is obviously deducible in restricted arithmetic. Therefore, the formula

$$A \rightarrow B$$

is also deducible in restricted arithmetic.

THEOREM. *If to the axioms of restricted arithmetic there are adjoined as axioms arbitrary formulae $A_1, \ldots, A_m$ of arithmetic, which do not contain predicate variables, then either the calculus obtained is inconsistent or the axiom of complete induction is not deducible in it.*

Note that we can assume that the formulae $A_1, \ldots, A_m$ do not contain propositional variables. In fact, suppose the formula A_i contains the propositional variable A. We write it in the form $A_i(A)$. It can be shown that the formula $A_i(A)$ is deductively equivalent to the formula

$$A_i(T) \mathbin{\&} A_i(F),$$

where T is an arbitrary true formula and F is an arbitrary false formula. This is obvious in one direction [the deducibility of $A_i(A)$ implies the deducibility of $A_i(T) \mathbin{\&} A_i(F)$]. The converse is easily proved, starting from elementary formulae, by induction on the construction of formulae. [If $A(A)$ is A, then $T \mathbin{\&} F$ is a false formula; consequently, an arbitrary formula is deducible from it—in particular, A. It is next proved that if the assertion is valid for the formulae A, A_1, A_2, then it is valid for

$$\bar{A},\ A_1 \mathbin{\&} A_2,\ A_1 \vee A_2,\ (x)A(x),\ (\exists x)A(x).]$$

So, we can assume that the axioms $A_1, \ldots, A_m$ do not contain propositional variables. In exactly the same way, we can assume that there are no free object variables in these axioms since the formulae $A(x)$ and $(x)A(x)$ are deductively equivalent in restricted arithmetic. We shall assume that the axiom of complete induction is deduced from the axioms of restricted arithmetic and from the axioms $A_1, \ldots, A_m$. Then, on the basis of the deduction theorem in restricted arithmetic, the formula

$$A_1 \mathbin{\&} A_2 \mathbin{\&} \ldots \mathbin{\&} A_m \rightarrow B \tag{*}$$

is provable, where B is the axiom of complete induction. We write this formula in the reduced form, also replacing the formulae A_i and B by their

reduced forms; we keep the same notation for the reduced forms of these formulae. Then the formula (*), above, takes on the following form:

$$\bar{A_1} \vee \bar{A_2} \vee \ldots \vee \bar{A_m} \vee B. \tag{1}$$

On the basis of Theorem 2, §9, formula (1) is regular. We shall prove that *in this case the formula*

$$\overline{\bar{A_1} \vee \ldots \vee \bar{A_m}} \tag{2}$$

is also regular.

Suppose $K_0, \ldots, K_q$ is a regularity series of formula (1) such that K_q coincides with formula (1). As we saw above, formula B has the form

$$\bar{A}(0) \vee (\exists x)(A(x) \,\&\, \bar{A}(x')) \vee A(y).$$

We can assume that the variable y does not occur in formula (2). Therefore, if we replace y by the numeral $0^{(n+1)}$ in each formula K_i, we obtain the regularity series

$$H_0, H_1, \ldots, H_q,$$

where H_q is the formula

$$\bar{A_1} \vee \ldots \vee \bar{A_m} \vee \bar{A}(0) \vee (\exists x)(A(x) \,\&\, \bar{A}(x')) \vee A(0^{(n+1)}).$$

In order to prove the regularity of (2), we shall show that we can, without loss of regularity, remove from all exterior factors of an arbitrary formula H_i all summands which occur in B, or which arise from B as the result of applying operations 1, 2, 3. We shall first prove that this assertion is valid for H_0. All the exterior factors of H_0 are elementary regular and can be written in the form

$$B_0' \vee B_0'' \vee A_0 \vee A_0',$$

where B_0' is the sum of primitive summands arising from B; B_0'' is the sum of the remaining summands arising from B; A_0 is the sum of primitive summands not appearing in B_0'; A_0' is the sum of all the remaining summands. Then the formula $B_0' \vee A_0$ is primitively true in the strong sense. We consider the formula B_0' in more detail. This formula contains a term arising from B. But, as we saw above (in §11), it is possible to apply only the one operation 2 to B and to all members arising from it, where the operation 2 is always performed on the summand

$$(\exists x)(A(x) \,\&\, \bar{A}(x')).$$

All the isolated members under these operations have the form

$$A(t) \,\&\, \bar{A}(t'),$$

where t is an arbitrary term. In this case, the formula B_0' has the form

$$\bar{A}(0) \vee A(t_1) \,\&\, \bar{A}(t_1') \vee \ldots \vee A(t_k) \,\&\, \bar{A}(t_k') \vee A(0^{(n+1)}).$$

Since we chose the numeral $0^{(n+1)}$ arbitrarily, it can be assumed that $k \leqslant n$ holds for all exterior factors of the formula H_0. But if for some terms k turned out to be less than n, then it is possible to apply operation 2 several

times more to the term $B_0' \vee B_0''$ obtained in order that the number of summands $A(t_i) \,\&\, \bar{A}(t_1')$ in the isolated members be exactly equal to n. Then the formula B_0' will finally take on the form

$$\bar{A}(0) \vee A(t_1) \,\&\, \bar{A}(t_1') \vee \ldots \vee A(t_n) \,\&\, \bar{A}(t_n') \vee A(0^{(n+1)}).$$

Since every formula $B_0' \vee A_0$ is primitively true in the strong sense, it is deducible in restricted arithmetic. We shall show that in this case the formula A_0 is deducible in restricted arithmetic. If we perform a substitution in the formula $B_0' \vee A_0$ replacing the predicate $A(t)$ by an arbitrary formula $X(t)$, then the term A_0 does not change. In fact, the members of this term are the terms of the exterior factor H_0 and arise from the term $A_1^- \vee \ldots \vee A_m^-$ of formula (1) as the result of applying operations 1, 2, 3. But, by assumption, the formula $A_1^- \vee \ldots \vee A_m^-$ does not contain predicate variables. Therefore, the terms arising from it of the exterior factors of all the formulae H_i also do not contain any predicate variables, in particular the predicate $A(\,)$. Thus, after the indicated substitution, the formula $B_0' \vee A_0$ goes over into the formula

$$\bar{X}(0) \vee X(t_1) \,\&\, \bar{X}(t_1') \vee \ldots \vee X(t_n) \,\&\, \bar{X}(t_n') \vee X(0^{(n+1)}) \vee A_0. \tag{3}$$

We now choose for $X(t)$ the formula

$$\begin{gathered} t = 0 \vee \Big(\sum_{i=1}^{n}(t_i = 0)\Big) \,\&\, t = 0' \vee \\ \vee \Big(\sum_{i=1}^{n}(t_i = 0)\Big) \,\&\, \Big(\sum_{i=1}^{n}(t_i = 0')\Big) \,\&\, t = 0'' \vee \ldots \\ \ldots \vee \Big(\sum_{i=1}^{n}(t_i = 0)\Big) \,\&\, \ldots \,\&\, \Big(\sum_{i=1}^{n}(t_i = 0^{(n-1)})\Big) \,\&\, t = 0^{(n)}, \end{gathered}$$

where the sign Σ denotes the logical sum.

It is quite easy to see that the formulae $X(0)$ and $\bar{X}(0^{(n+1)})$ are deducible in restricted arithmetic. We shall show that all the formulae

$$\overline{\bar{X}(t_j) \,\&\, \bar{X}(t_j')},\ j = 1, 2, \ldots, n$$

are also deducible in restricted arithmetic.

We consider the formula $X(t_j)$, i.e.

$$\begin{gathered} t_j = 0 \vee \Big(\sum_{i=1}^{n}(t_i = 0)\Big) \,\&\, t_j = 0' \vee \ldots \\ \ldots \vee \Big(\sum_{i=1}^{n}(t_i = 0)\Big) \,\&\, \ldots \,\&\, \Big(\sum_{i=1}^{n}(t_i = 0^{(n-1)})\Big) \,\&\, t_j = 0^{(n)}. \end{gathered} \tag{4}$$

The formula $\bar{X}(t_j')$ is obviously equivalent to the formula

$$\begin{gathered} \overline{t_j' = 0} \,\&\, \Big(\prod_{i=1}^{n}\overline{(t_i = 0)} \vee \overline{t_j' = 0'}\Big) \,\&\, \ldots \\ \ldots \,\&\, \Big(\prod_{i=1}^{n}\overline{(t_i = 0)} \vee \ldots \vee \prod_{i=1}^{n}\overline{(t_i = 0^{(n-1)})} \vee \overline{(t_j' = 0^{(n)})}\Big), \end{gathered} \tag{5}$$

where the sign Π denotes the logical product.

We now deduce some formal consequences from formulae (4) and (5) which we obtain with the aid of all true formulae and the rules of restricted arithmetic. The formula

$$\sum_{i=1}^{n}(t_i = 0) \vee \Big(\sum_{i=1}^{n}(t_i = 0)\Big) \mathbin{\&} t_j = 0' \vee \ldots$$
$$\ldots \vee \Big(\sum_{i=1}^{n}(t_i = 0)\Big) \mathbin{\&} \ldots \mathbin{\&} \Big(\sum_{i=1}^{n}(t_i = 0^{(n-1)})\Big) \mathbin{\&} t_j = 0^{(n)} \quad (6)$$

is deduced from formula (4) in this way.

Formula (6) is obtained from formula (4) by the adjunction to it of the new terms $t_i = 0$, $i \neq j$. The formula

$$\sum_{i=1}^{n}(t_i = 0)$$

is deducible from formula (6). In fact, we consider the true formula of the propositional calculus:

$$A_1 \vee A_1 \mathbin{\&} B_1 \vee \ldots \vee A_1 \mathbin{\&} B_{n-1} \to A_1.$$

Replacing A_1 by

$$\sum_{i=1}^{n}(t_i = 0)$$

and B_1 by the corresponding expression with this one in order to obtain formula (6) in the antecedent, we obtain the true formula

$$\sum_{i=1}^{n}(t_i = 0) \vee \Big(\sum_{i=1}^{n}(t_i = 0)\Big) \mathbin{\&} t_j = 0' \vee \ldots$$
$$\ldots \vee \Big(\sum_{i=1}^{n}(t_i = 0)\Big) \mathbin{\&} \ldots \mathbin{\&} \Big(\sum_{i=1}^{n}(t_i = 0^{(n-1)})\Big) \mathbin{\&}$$
$$\mathbin{\&} t_j = 0^{(n)} \to \sum_{i=1}^{n}(t_i = 0).$$

Applying the rule of inference, we obtain that the formula

$$\prod_{i=1}^{n}(t_i = 0)$$

is deducible from formula (6) and consequently from formula (4) also. The formula

$$\prod_{i=1}^{n}\overline{(t_i = 0)} \vee \overline{(t_j' = 0')}$$

is deducible from formula (5). The first term of this formula is equivalent to the formula

$$\overline{\sum_{i=0}^{n}(t_i = 0)}.$$

In this case, the formula

$$\sum_{i=1}^{n}(t_i = 0) \to \overline{t_j' = 0'}$$

is deducible.

Applying the rule of inference, we find that the formula

$$\overline{t_j' = 0'}$$

is deducible from formulae (4) and (5).

The formula

$$\overline{(t_j' = 0')} \rightarrow \overline{(t_j = 0)}$$

is deducible in restricted arithmetic. Therefore, the formula

$$\overline{t_j = 0}$$

is also deducible from formulae (4) and (5). From it and from formula (4), we can deduce the formula

$$\Big(\sum_{i=1}^{n}(t_i = 0)\Big) \,\&\, t_j = 0' \vee \ldots$$
$$\ldots \vee \Big(\sum_{i=1}^{n}(t_i = 0)\Big) \,\&\, \ldots \,\&\, \Big(\sum_{i=1}^{n}(t_i = 0^{(n-1)})\Big) \,\&\, t_j = 0^{(n)}.$$

For the deduction, it is sufficient to discard the false term $t_j = 0$ in formula (4). Further, taking the factor $\sum_{i=1}^{n}(t_i = 0)$ outside the brackets in the formula obtained, we obtain the formula

$$\Big(\sum_{i=1}^{n}(t_i = 0)\Big) \,\&\, t_j = 0' \vee \Big(\sum_{i=1}^{n}(t_i = 0')\Big) \,\&\, t_j = 0'' \vee \ldots$$
$$\ldots \vee \Big(\sum_{i=1}^{n}(t_i = 0')\Big) \,\&\, \ldots \,\&\, \Big)\sum_{i=1}^{n}(t_i = 0^{(n-1)})\Big) \,\&\, t_j = 0^{(n)}$$

and, discarding the first factor, we have

$$(t_j = 0') \vee \Big(\sum_{i=1}^{n}(t_i = 0')\Big) \,\&\, t_j = 0'' \vee \ldots$$
$$\ldots \vee \Big(\sum_{i=1}^{n}(t_i = 0')\Big) \,\&\, \ldots \,\&\, \Big(\sum_{i=1}^{n}(t_i = 0^{(n-1)})\Big) \,\&\, t_j = 0^{(n)}. \tag{7}$$

From formula (7), we can further deduce the formula

$$\sum_{i=1}^{n}(t_i = 0')$$

by exactly the same method as the formula $\sum_{i=1}^{n}(t_i = 0)$ was deduced above. We can deduce the formula

$$\prod_{i=1}^{n}\overline{(t_i = 0')} \vee \overline{t_j' = 0''}$$

from formula (5). In fact, the formula

$$\prod_{i=1}^{n}\overline{(t_i = 0)} \vee \prod_{i=1}^{n}\overline{(t_i = 0')} \vee \overline{t_j' = 0''}$$

is directly deducible from (5). The first term can be discarded since it is false. In fact, $\overline{\prod_{i=1}^{n}(t_i = 0)}$ coincides with $\sum_{i=1}^{n}\overline{(t_i = 0)}$ and, as we have seen, the formula $\sum_{i=1}^{n}(t_i = 0)$ is deducible from (4) and (5). From this, in the same way as we deduced the formula $\overline{t_j = 0}$ above, we deduce the formula

$$\overline{t_j = 0'}.$$

This done, discarding the first (false) term from formula (7), we obtain the formula

$$\begin{aligned}\Big(\sum_{i=1}^{n}(t_i = 0')\Big) \,\&\, t_j = 0'' \vee \ldots \\ \ldots \vee \Big(\sum_{i=1}^{n}(t_i = 0')\Big) \,\&\, \ldots \,\&\, \Big(\sum_{i=1}^{n}(t_i = 0^{(n-1)})\Big) \,\&\, t_j = 0^{(n)}.\end{aligned} \tag{8}$$

Continuing the same line of reasoning further, we obtain the following formulae which are deducible from formulae (4) and (5):

$$\begin{aligned}\sum_{i=1}^{n}(t_i = 0), \overline{t_j = 0}; \sum_{i=1}^{n}(t_i = 0'), \overline{t_j = 0'}; \ldots \\ \ldots; \sum_{i=1}^{n}(t_i = 0^{(n-1)}), \overline{t_j = 0^{(n-1)}}.\end{aligned}$$

The following formulae are obviously deducible from them:

$$\sum_{i \neq j}(t_i = 0); \sum_{i \neq j}(t_i = 0'); \ldots ; \sum_{i \neq j}(t_i = 0^{(n-1)}). \tag{9}$$

The product of all the formulae (9) is also deducible from (4) and (5):

$$\Big(\sum_{i \neq j}(t_i = 0)\Big) \,\&\, \Big(\sum_{i \neq j}(t_i = 0')\Big) \,\&\, \ldots \,\&\, \Big(\sum_{i \neq j}(t_i = 0^{(n-1)})\Big). \tag{10}$$

We consider now the disjunctive normal form of formula (10). It is also deducible from (4) and (5) and has the form

$$\sum_{i_1 \neq j, \ldots, i_n \neq j}(t_{i_1} = 0) \,\&\, (t_{i_2} = 0') \,\&\, \ldots \,\&\, (t_{i_n} = 0^{(n-1)}). \tag{11}$$

However, every term of this sum is false in restricted arithmetic. In fact, every index $i_1, \ldots, i_n$ can take on only $n - 1$ values:

$$1, 2, \ldots, j - 1, j + 1, \ldots, n.$$

Therefore, in every term of formula (11) at least two of the indices $i_1, i_2, \ldots, i_n$ take on the same value. Suppose the indices i_r and i_s of some term take on the value r. In this case, there occur two factors in this term:

$$t_r = 0^{(p-1)} \text{ and } t_r = 0^{(q-1)}, p \neq q.$$

The formula

$$(t_r = 0^{(p-1)}) \,\&\, (t_r = 0^{(q-1)}) \to 0^{(p-1)} = 0^{(q-1)}$$

is deducible in restricted arithmetic. But the formula

$$0^{(p-1)} = 0^{(q-1)}$$

is false since $p \neq q$; consequently the formula

$$(t_r = 0^{(p-1)}) \mathbin{\&} (t_r = 0^{(q-1)})$$

is also false. But then the entire product

$$(t_{i_1} = 0) \mathbin{\&} (t_{i_2} = 0') \mathbin{\&} \ldots \mathbin{\&} (t_{i_n} = 0^{(n-1)})$$

is also false. If we denote formula (11) by the letter C, then the formula $\overline{C}$ is deducible in restricted arithmetic. Since C is deducible from true formulae of restricted arithmetic and formulae (4) and (5), it is consequently deducible from the formulae $X(t_j)$ and $X(t_j')$ which are equivalent to (4) and (5). In this deduction of C, we do not substitute in the variables of the initial formula and we do not bind them with quantifiers. Hence, applying the deduction theorem, we see that the formula

$$X(t_j) \mathbin{\&} \overline{X}(t_j') \rightarrow C,$$

and consequently also the formula

$$\overline{C} \rightarrow \overline{X(t_j) \mathbin{\&} \overline{X}(t_j')}$$

is deducible in the extended predicate calculus. But since formula $\overline{C}$ is deducible in restricted arithmetic, the formula

$$\overline{X(t_j) \mathbin{\&} \overline{X}(t_j')}$$

is also deducible in it. So, for every index j from 1 to n, the term

$$X(t_j) \mathbin{\&} \overline{X}(t_j')$$

of formula (3) is false in restricted arithmetic. Since, according to what was proved above, the formulae $\overline{X}(0)$ and $X(0^{(n+1)})$ are also false in restricted arithmetic, all the terms of formula (3), with the exception of A_0, are false; consequently, the formula B_0' is not deducible in our calculus. But since formula (3) is deducible in restricted arithmetic, A_0 is deducible in it also. So, if we eliminate the terms B_0' and B_0'' in every exterior factor of the formula H_0, the remaining terms form formulae which are deducible in restricted arithmetic. We have thus shown that by eliminating from the exterior factors of H_0 the terms arising from B we obtain formulae which are deducible in restricted arithmetic—and consequently are regular formulae.

We shall assume that this assertion is valid for the formula H_{i-1} and show that it is then valid for H_i. Let G_i be an arbitrary exterior factor of H_i and let G_i' be the formula obtained from G_i after eliminating the terms arising from B. If the operation by means of which H_{i-1} is obtained from H_i is performed on terms arising from B, then the factor G_i' can also be obtained by the elimination of terms, arising from B, from some exterior factor G_{i-1} (arising from G_i) of the formula H_{i-1}. Therefore, according to the induction assumption, G_i is a regular formula. Suppose now that the operation carrying H_i into H_{i-1} is performed on terms not arising from B. We eliminate from G_i and G_{i-1} terms arising from B, and we denote the formulae obtained by

G_i' and G_{i-1}' (in the case when operation 3 is applied to the factor G_i, we shall understand by G_{i-1} the product of all exterior factors of the formula H_{i-1} arising from G_i). Obviously, the formula G_{i-1}' is obtained from G_i' by the same operation as is the formula G_{i-1} from G_i. By virtue of the induction assumption, the formula G_{i-1}' is regular. But then G_i is also regular. So, we have proved that formulae obtained as the result of eliminating from the exterior factors of H_i terms arising from B remain regular. In this case, this assertion is also valid for the formula H_q consisting of the single exterior factor

$$A_1^- \vee A_2^- \vee \ldots \vee A_m^- \vee B.$$

We have thus proved that the formula $A_1^- \vee \ldots \vee A_m^-$ is regular. But then, by virtue of property 4, §5, this formula is deducible in restricted arithmetic. But since A_i^- is equivalent to $\bar{A}_i$, the formula

$$\bar{A}_1 \vee \ldots \vee \bar{A}_m$$

is also deducible in restricted arithmetic. But this formula is equivalent to the formula

$$\overline{A_1 \mathbin{\&} \ldots \mathbin{\&} A_m}.$$

We have arrived at a contradiction as both the formulae

$$A_1 \mathbin{\&} \ldots \mathbin{\&} A_m \text{ and } \overline{A_1 \mathbin{\&} \ldots \mathbin{\&} A_m}$$

turn out to be true in our calculus. Thus, the assumption that the axiom of complete induction is deducible in the calculus obtained by the adjunction of the formulae $A_1, \ldots, A_m$ to the axioms of restricted arithmetic, makes this calculus inconsistent and so the theorem is proved.

INDEX OF SYMBOLS

in order of their first appearance

GENERAL INDEX